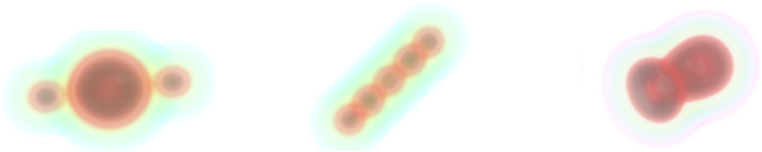

Nuclear clustering within the beyond RMF framework

P. Marević[1,2], J.-P. Ebran[1], E. Khan[2], T. Nikšić[3], D. Vretenar[3]

[1]CEA, DAM, DIF, France

[2]IPN Orsay, France

[3]University of Zagreb, Croatia

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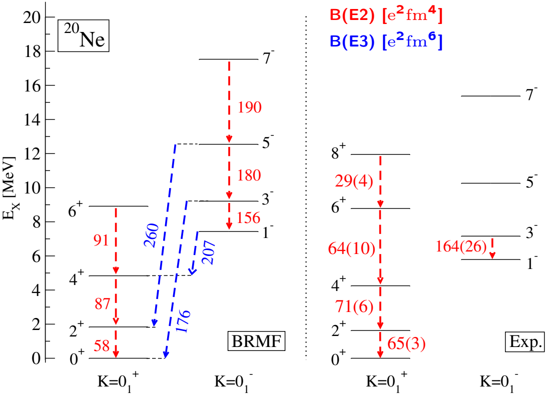
A sneak peek: ^{20}Ne spectroscopy



Outline of the talk

1 Outline of the talk

Theoretical framework

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Results

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Results

Systematics of neon isotopes

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Outline of the talk

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- **Systematics of neon isotopes**
- **Structure and clusters in ^{20}Ne**

1

Outline of the talk

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- **Preliminary: ^{12}C isotope**

1

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Conclusion and outlook

Theoretical framework

2

Theoretical framework
Single-reference (mean-field) level

Theoretical framework
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- NEDFs as global theoretical framework

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  - DD-PC1 functional [T. Nikšić *et al.* PRC 78, 034318 (2008).]

$$\mathcal{L} = \bar{\psi}(i\gamma \cdot \partial - m)\psi - \frac{1}{2}\alpha_S(\hat{\rho})(\bar{\psi}\psi)(\bar{\psi}\psi) - \frac{1}{2}\alpha_V(\hat{\rho})(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)$$
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3

Theoretical framework
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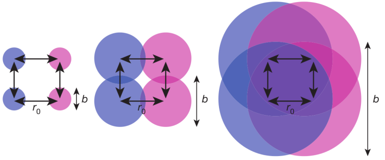
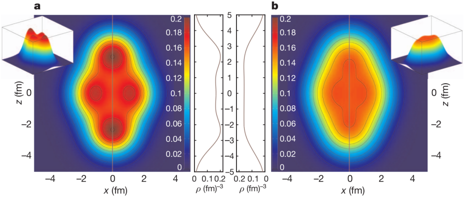


$\alpha < 1$ $\alpha \approx 1$ $\alpha > 1$



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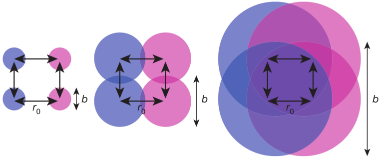

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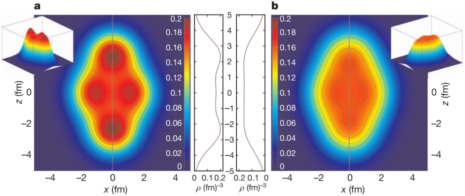

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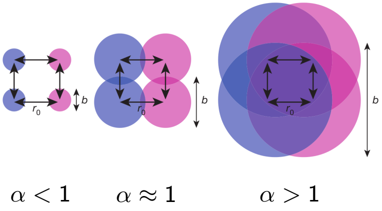

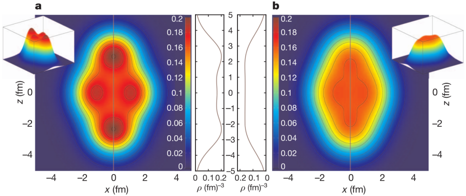

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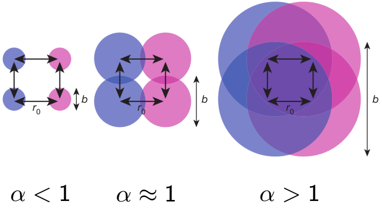

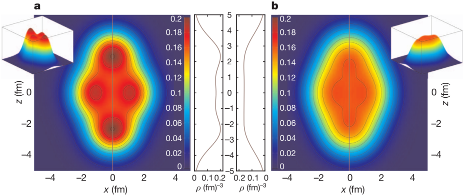

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Beyond RMF description of light nuclei

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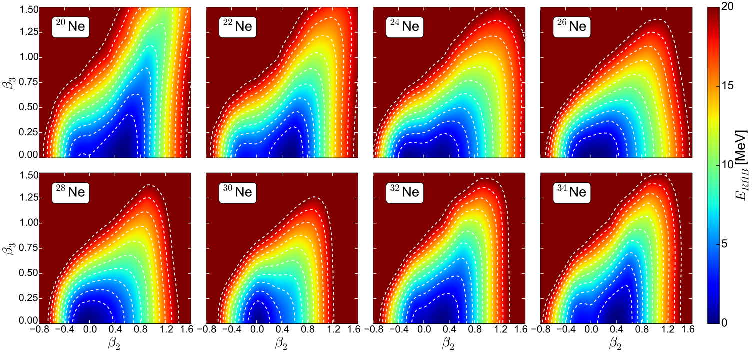
Neon isotopes: mean-field energies

5

Beyond RMF description of light nuclei
Neon isotopes: mean-field energies



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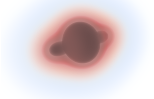

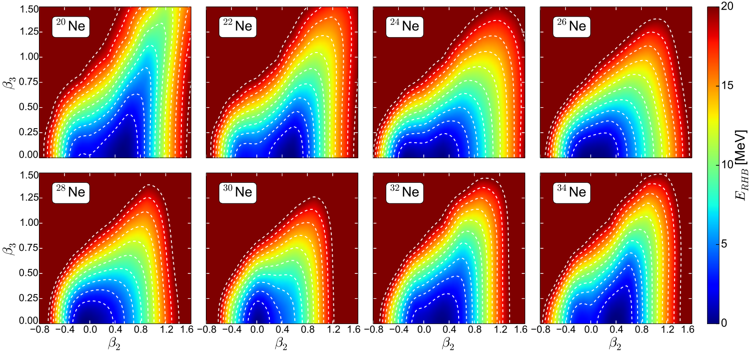

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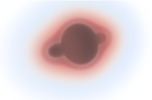

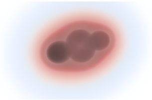

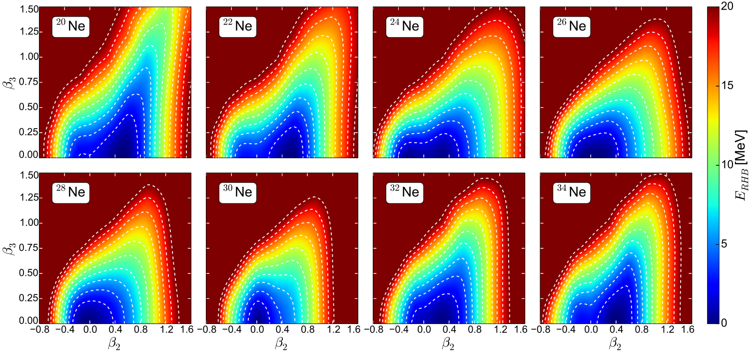

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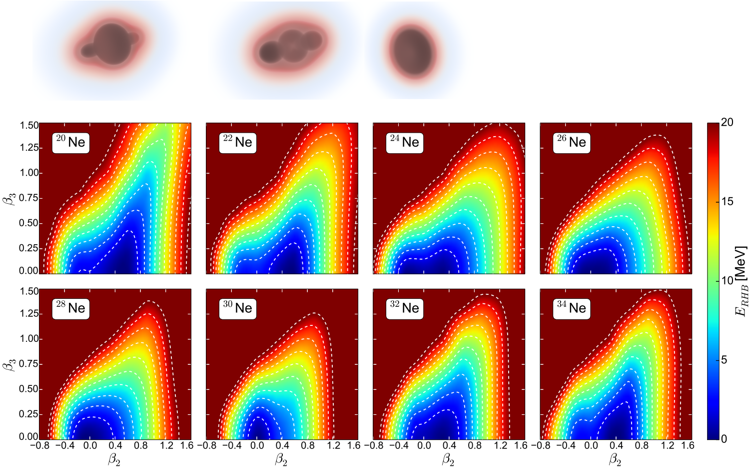

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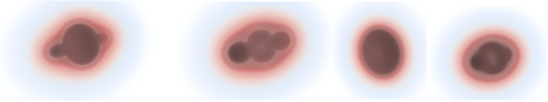

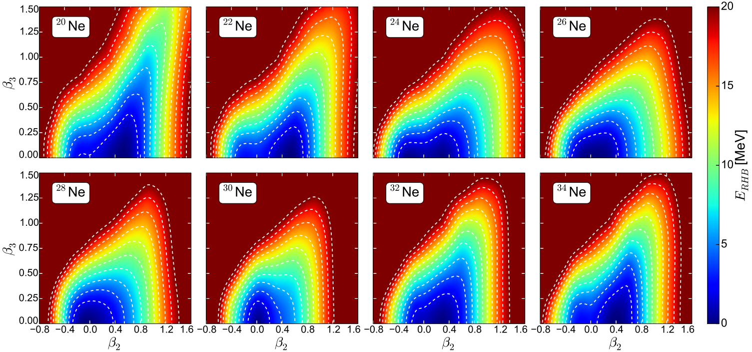

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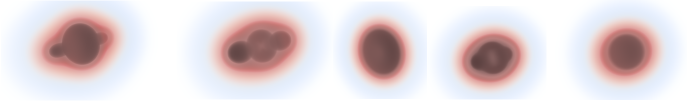

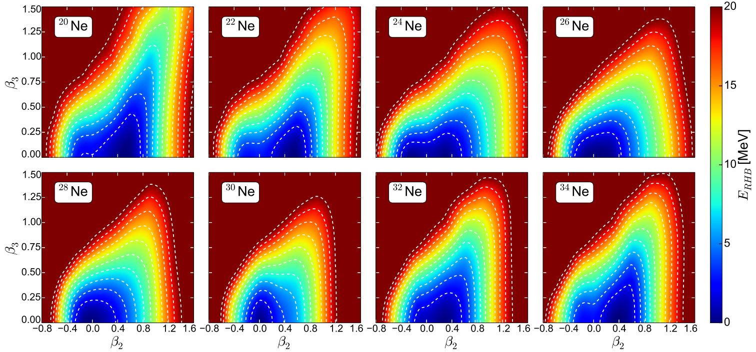

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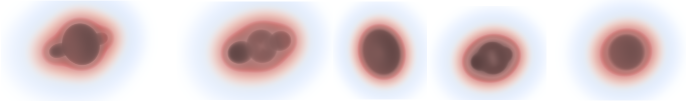

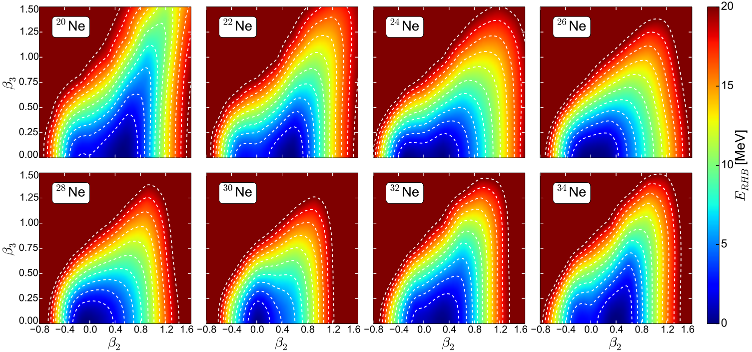

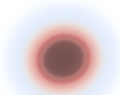

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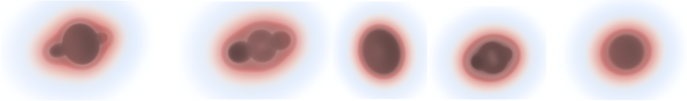

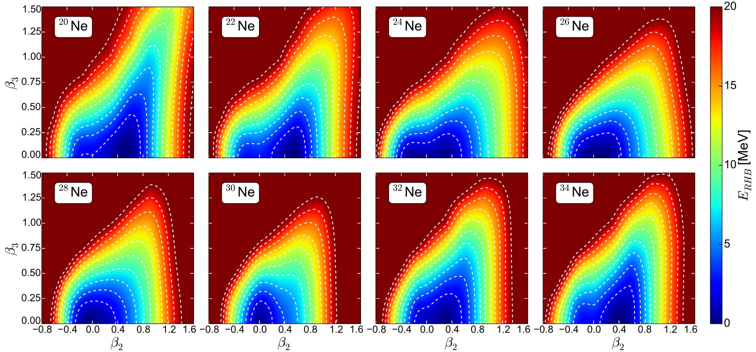

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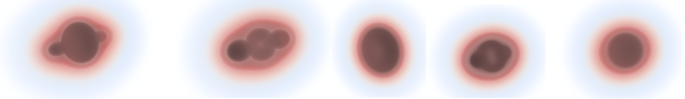

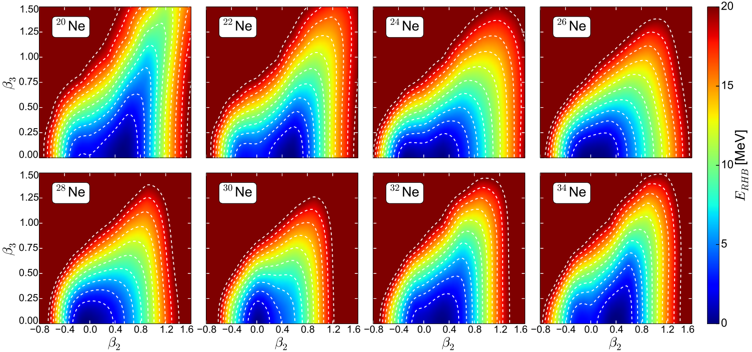

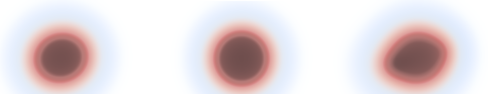

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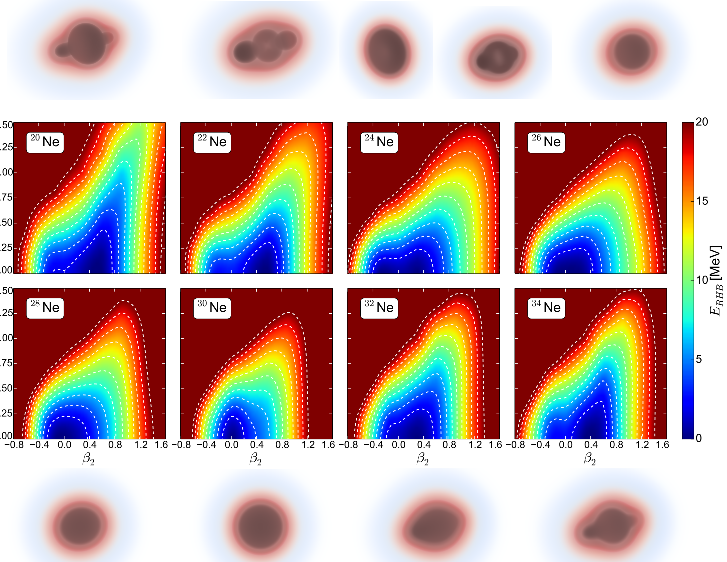
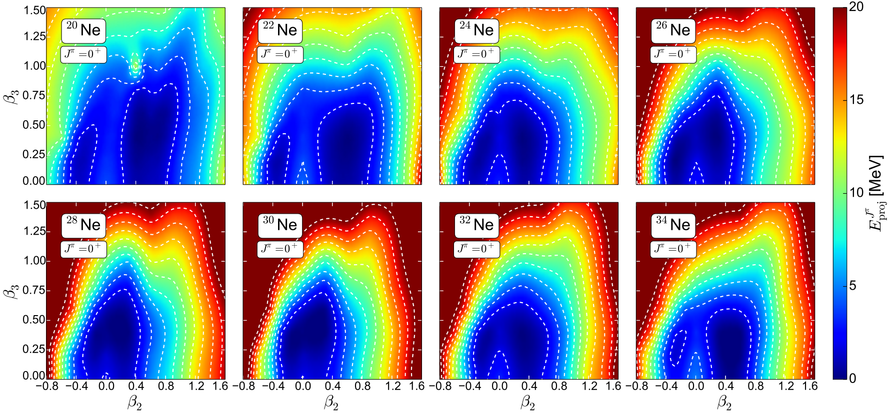


Beyond RMF description of light nuclei
Neon isotopes: symmetry-restored energies



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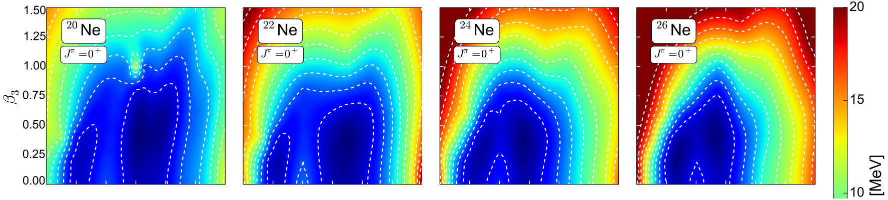


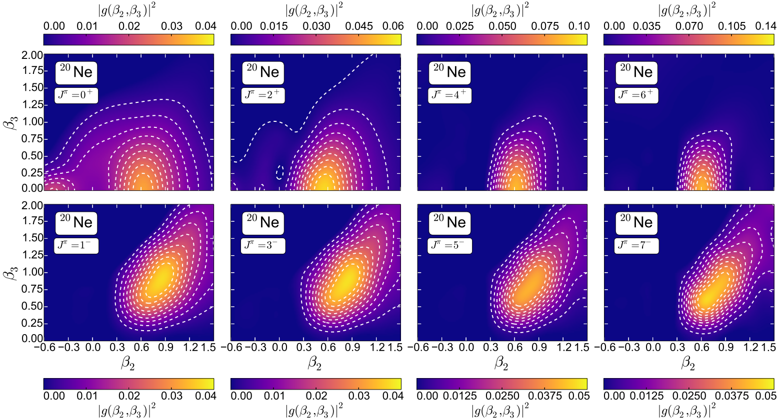
PHYSICAL REVIEW C **97**, 024334 (2018)

Quadrupole and octupole collectivity and cluster structures in neon isotopes

P. Marević,[1,2] J.-P. Ebran,[1] E. Khan,[2] T. Nikšić,[3] and D. Vretenar[3]
[1]*CEA, DAM, DIF, F-91297 Arpajon, France*
[2]*Institut de Physique Nucléaire, Université Paris-Sud, IN2P3-CNRS, Université Paris-Saclay, F-91406 Orsay Cedex, France*
[3]*Department of Physics, Faculty of Science, University of Zagreb, Bijenička c. 32, 10000 Zagreb, Croatia*

Beyond RMF description of light nuclei
Neon isotopes: self-conjugate ^{20}Ne nucleus

Collective wave functions:



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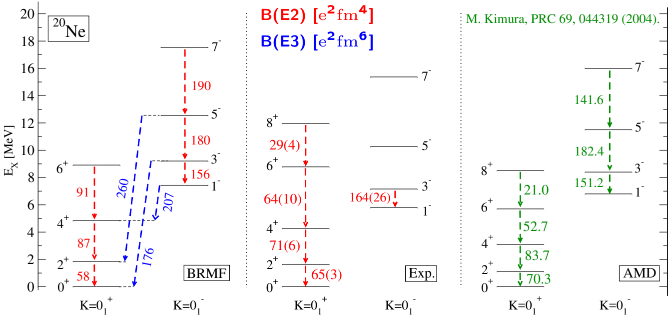

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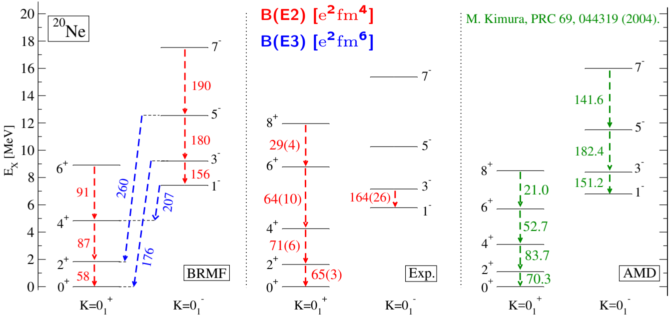

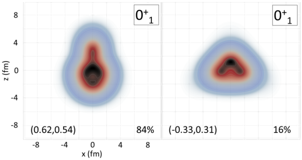

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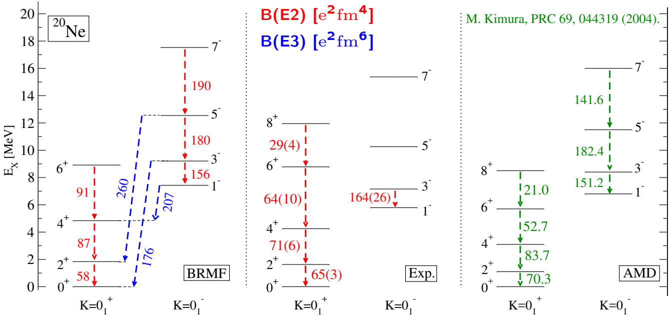
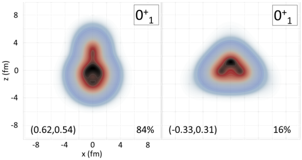
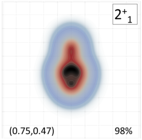
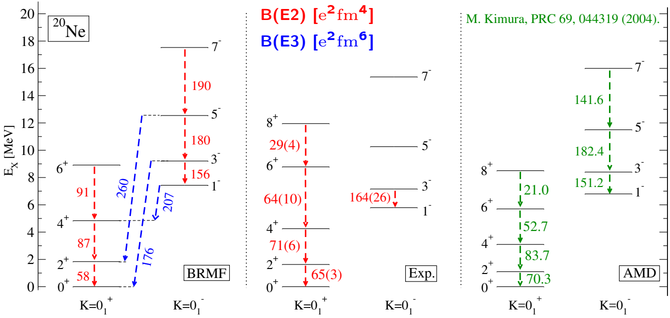


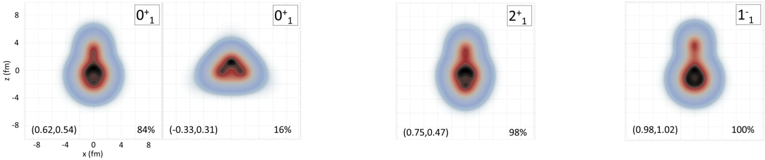




8

Beyond RMF description of light nuclei
Neon isotopes: self-conjugate ^{20}Ne nucleus





Beyond RMF description of light nuclei

Structure of ^{12}C isotope (preliminary)

9

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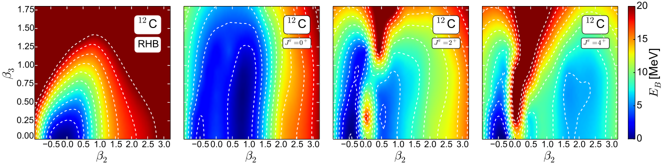

9

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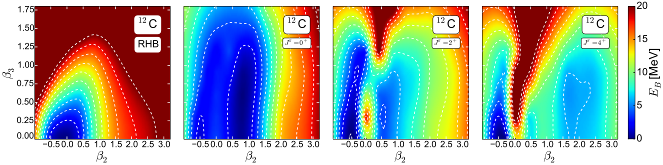

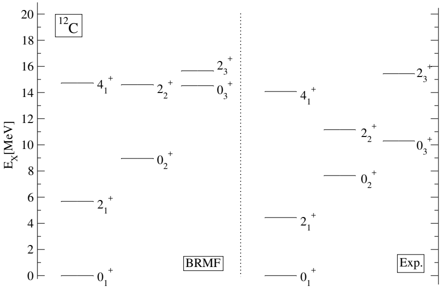

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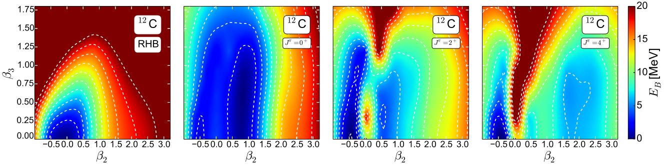

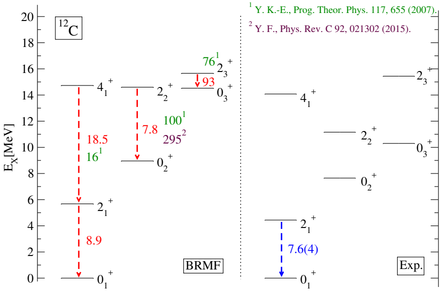

Beyond RMF description of light nuclei

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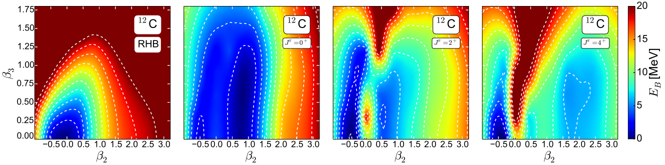

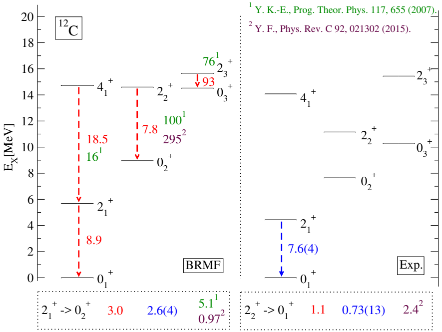

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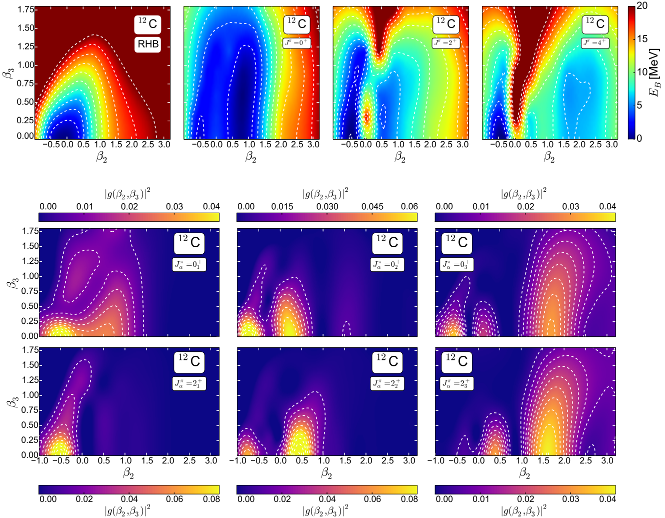

Conclusion

A wrap-up

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- REDF as powerful tool for nuclear structure calculations
  - mean-field description of ground-state properties
  - clustering due to the depth of confining potential
  - quantitative analysis on a beyond mean-field level

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  - systematics of Ne isotopic chain **[P. M. *et al.*, PRC 97, 024334 (2018).]**
  - collective properties and cluster structures in ^{20}Ne
  - promising preliminary results for ^{12}C
- towards unified description of quantum-liquid and cluster states
  - calculation of (in)elastic form factors and charge radii
  - structure of heavy nuclei
  - inclusion of the triaxial degree of freedom

Thank you for your attention!

Theoretical framework - Backup

- constrained RHB solutions as BMF input: $|\phi(q_j)\rangle$, $q \equiv (\beta_2, \beta_3)$
- angular momentum projection

$$\hat{P}^J_{MK} = \frac{2J+1}{8\pi^2} \int d\Omega {D^J}^*_{MK}(\Omega)\hat{R}(\Omega)$$

$$\Omega = (\alpha, \beta, \gamma), \qquad D^J_{MK}(\Omega) = e^{-iM\alpha} d^J_{MK}(\beta) e^{-iK\gamma}, \qquad \hat{R} = e^{-i\alpha \hat{J}_z} e^{-i\beta \hat{J}_y} e^{-i\gamma \hat{J}_z}$$

- particle number projection with Fomenko expansion
- linear combination of symmetry-projected RHB states

$$\underbrace{|JNZM\pi;\alpha\rangle}_{\text{collective state}} = \sum_j \sum_K \underbrace{f^{JK\pi}_\alpha(q_j)}_{\text{weight function}} \underbrace{\hat{P}^J_{MK}\hat{P}^\pi \hat{P}^N \hat{P}^Z}_{\text{projectors}} \underbrace{|\phi(q_j)\rangle}_{\text{RHB state}}$$

Theoretical framework - Backup

- preserved symmetries
  - time-reversal symmetry (even-even nuclei)
  - axial symmetry ($\hat{J}_z \left|\phi(q_j)\right\rangle = 0$, $|JNZM\pi;\alpha\rangle \rightarrow |JNZ\pi;\alpha\rangle$)
  - simplex-x symmetry ($\hat{P}e^{-i\pi\hat{J}_x}$)
- variational equation determines weight functions $f_\alpha^{J\pi}$

$$\delta E^{J\pi} = \delta \frac{\langle JNZ\pi;\alpha|\,\hat{H}\,|JNZ\pi;\alpha\rangle}{\langle JNZ\pi;\alpha|JNZ\pi;\alpha\rangle} = 0$$

- generator coordinate method (GCM) framework [Hill, Wheeler, PR 89, 1102 (1953).; L. Lathouwers, AoP 102, 347 (1976).; Ring, Schuck, *Nuclear Many-Body Problem*]

ACKNOWLEDGMENT

The essential part of the work which led to this article was done on Christmas Day 1974. I therefore want to thank Santa Claus for inspiration. I am grateful to Prof. Per-Olov Löwdin for being able to benefit from the scientific atmosphere at the Uppsala Quantum Chemistry Group and to Profs. M. Bouten and P. Van Leuven for valuable comments.

Theoretical framework - Backup

- discretized Hill-Wheeler-Griffin (HWG) equation

$$\sum_j \left[\underbrace{\mathcal{H}^{J\pi}(q_i, q_j)}_{\text{Hamiltonian kernel}} - E_\alpha^{J\pi} \underbrace{\mathcal{N}^{J\pi}(q_i, q_j)}_{\text{norm kernel}} \right] f_\alpha^{J\pi}(q_j) = 0$$

$$\mathcal{O}^{J\pi}(q_i, q_j) = \frac{2J+1}{2}\delta_{M0}\delta_{K0}\int_0^\pi d\beta \sin(\beta) d_{00}^{J*}(\beta) \langle \Phi(q_i)| \hat{O} e^{-i\beta \hat{J}_x} \hat{P}^\pi |\Phi(q_j)\rangle$$

- one rewrites the HWG equation as an ordinary eigenvalue problem

$$\sum_j \tilde{\mathcal{H}}^{J\pi}(q_i, q_j) g_\alpha^{J\pi}(q_j) = E_\alpha^{J\pi} g_\alpha^{J\pi}(q_i)$$

where modified Hamiltonian kernel reads

$$\tilde{\mathcal{H}}^{J\pi}(q_i, q_j) = \sum_{k,l} \left[(\mathcal{N}^{J\pi})^{-1/2}(q_i, q_k) \mathcal{H}^{J\pi}(q_k, q_l) (\mathcal{N}^{J\pi})^{-1/2}(q_l, q_j) \right]$$

and collective wave functions read

$$g_\alpha^{J\pi}(q_i) = \sum_j (\mathcal{N}^{J\pi})^{1/2}(q_i, q_j) f_\alpha^{J\pi}(q_j)$$

Theoretical framework - Backup

- HWG equation is solved by diagonalizing the norm kernel

$$\sum_j \mathcal{N}^{J\pi}(q_i, q_j) u_k^{J\pi}(q_j) = n_k^{J\pi} u_k^{J\pi}(q_i)$$

and eliminating from the basis states with $n_k^{J\pi} < \zeta$

- the collective Hamiltonian is built from the remaining states

$$\mathcal{H}_{kl}^{J\pi c} = \frac{1}{\sqrt{n_k}} \frac{1}{\sqrt{n_l}} \sum_{i,j} u_k^{J\pi}(q_i) \tilde{\mathcal{H}}^{J\pi}(q_i, q_j) u_l^{J\pi}(q_j)$$

- and it is subsequently diagonalized for each (J, π)

$$\sum_l \mathcal{H}_{kl}^{J\pi c} g_l^{J\pi\alpha} = E_\alpha^{J\pi} g_k^{J\pi\alpha}$$

- calculation of physical observables via collective w.f. and weight functions

$$g_\alpha^{J\pi}(q_i) = \sum_k g_k^{J\pi\alpha} u_k^{J\pi}(q_i), \quad f_\alpha^{J\pi}(q_i) = \sum_k \frac{g_k^{J\pi\alpha}}{\sqrt{n_k^{J\pi}}} u_k^{J\pi}(q_i)$$