Astrophysical Resonant Reaction in Trojan Horse Method

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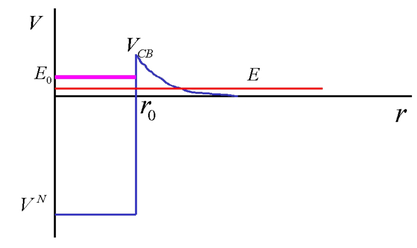
National Science Foundation Grant PHY-0354098
Research Experience for Undergraduates Summer Program 2004,
U.S. Department of Energy
Grant DE-FG02-93ER40773,
National Science Foundation Grant PHY-0140343

Nuclear astrophysical reactions occur at energies significantly below the Coulomb barrier. Therefore, cross-sections of astrophysical reactions are typically very small, and it is difficult, often impossible, to measure these cross-sections at astrophysically relevant energies.

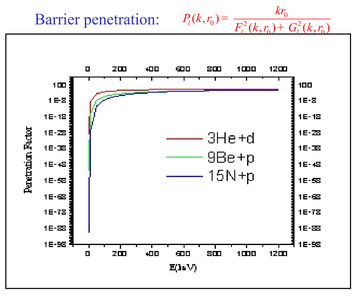
Because of the small cross sections of astrophysical reactions, different indirect techniques are being used. Among them is Trojan Horse, which allows one to measure cross sections of astrophysical rearrangement processes. In the Trojan Horse method, a projectile impinging on a target has high enough energy to overcome the Coulomb + centrifugal barrier. After penetration through the barrier the projectile virtually breaks into two particles, leaving one to interact with the target at astrophysically relevant energies. The fragment is virtual, and the extracted astrophysical cross section is half-off-energy-shell.
The goal of this project is to investigate the off-shell effects in the Trojan Horse method for the resonant reactions.

Resonance Width $\Gamma(E,r_0)=\frac{P_{l_0}(E,r_0)}{P_{l_0}(E_0,r_0)}\Gamma(E_0)$



Penetration Factors

Barrier penetration: $P_l(k,r_0)=\frac{kr_0}{F^2(k,r_0)+G^2(k,r_0)}$



Scattering and Reaction Amplitudes

On-shell resonance scattering amplitude: $(k^2/2\mu=E,\ k^2/2\mu=E)$

$$T(\boldsymbol{k}',\boldsymbol{k};E)=\langle \boldsymbol{k}'|V+VG^{(+)}(E)V|\boldsymbol{k}\rangle$$
$$=\int d\boldsymbol{r}\, e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{r}}V(r)+\int d\boldsymbol{r}'d\boldsymbol{r}\, e^{-i\boldsymbol{k}'\boldsymbol{r}'}V(r')G^{(+)C}(\boldsymbol{r}',\boldsymbol{r};E)V^C(r)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$
$$+\int d\boldsymbol{r}'d\boldsymbol{r}\, e^{-i\boldsymbol{k}'\boldsymbol{r}'}V^C(r')G^{(+)C}(\boldsymbol{r}',\boldsymbol{r};E)V^N(r)\psi_{\boldsymbol{k}}^{(+)C}(\boldsymbol{r})$$
$$+\int d\boldsymbol{r}'d\boldsymbol{r}\,\psi_{\boldsymbol{k}'}^{*(-)C}(\boldsymbol{r}')V^N(r')G^{(+)}(\boldsymbol{r}',\boldsymbol{r};E)V^N(r)\psi_{\boldsymbol{k}}^{(+)C}(\boldsymbol{r})$$

Resonant term (contains full Green function)

$$G^{(+)}V=G^{(+)}V^N+G^{(+)}V^C$$
$$=G^{(+)C}V^C+G^{(+)}V^N(1+G^{(+)C}V^C)$$
$$E=\frac{k^2}{2\mu}$$
$$(1+G^{(+)C}(E)V^C)|\boldsymbol{k}\rangle=\psi_{\boldsymbol{k}}^{C(+)}$$

If $E\neq\frac{k^2}{2\mu}$

then $(1+G^{(+)C}(E)V^C)|\boldsymbol{k}\rangle\neq\psi_{\boldsymbol{k}}^{C(+)}$

Green function partial wave expansion
(bound and continuum states form a complete set)

$$G^{(+)}(\boldsymbol{r}',\boldsymbol{r};E)=\sum_{n=0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\varphi_{nl}(r')Y_{lm}(\hat{\boldsymbol{r}}')\varphi_{nl}(r)Y_{lm}^*(\hat{\boldsymbol{r}})}{E+\varepsilon_{nl}}$$
$$+\sum_{m_0=-l_0}^{l_0}\frac{\psi_{nl_0}^{(r)}(r')Y_{l_0m}(\hat{\boldsymbol{r}}')\psi_{nl_0}^{(r)}(r)Y_{l_0m}^*(\hat{\boldsymbol{r}})}{E-E_{nl_0}^{(r)}}$$

Resonant term

$$+(4\pi)^2\sum_{l\neq l_0}Y_{lm}(\hat{\boldsymbol{r}}')Y_{lm}^*(\hat{\boldsymbol{r}})\int_0^{\infty}\frac{dp}{(2\pi)^3}p^2\frac{\psi_{lp}(r')\psi_{lp}(r)}{E-\frac{p^2}{2\mu}+i0}$$
$$+(4\pi)^2Y_{l_0m}(\hat{\boldsymbol{r}}')Y_{l_0m}^*(\hat{\boldsymbol{r}})\int\frac{dp}{(2\pi)^3}p^2\frac{\psi_{l_0p}(r')\psi_{l_0p}(r)}{E-\frac{p^2}{2\mu}+i0}$$

Integral along the contour in the complex plane

Resonant scattering amplitude

$$T^{(r)}(\boldsymbol{k}',\boldsymbol{k};E)=(4\pi)^2\sum_{m=-l_0}^{l_0}Y_{l_0m_0}(\hat{\boldsymbol{k}}')Y_{l_0m_0}^*(\hat{\boldsymbol{k}})\,e^{2i\sigma_{l_0}}\frac{W_{l_0}^2(k,k_r)}{E-E_{nl_0}^{(r)}}.$$

Form factor = resonance synthesis or decay amplitude

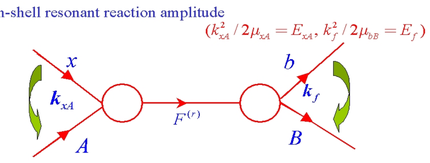
Form factor

$$W_{l_0}(k)=\int_0^{\infty}dr\,r^2\psi_{nl_0}^{(r)}(r)\,V^N(r)\psi_{l_0k}^C(r)$$
$$=(E_{nl_0}^{(r)}-E_k)\int_0^{\infty}dr\,r^2\psi_{nl_0}^{(r)}(r)\,\psi_{l_0k}^C(r)$$
$$W(k_r,k_r)=\lim_{k\to k_r}W(k,k_r)=-\frac{e^{i(\delta_{l_0}(k_r)-\sigma_{l_0}(k_r))}}{2\mu}\sqrt{\frac{\mu}{k_r}\Gamma}$$

On-shell resonant scattering amplitude

$$T^{(r)}(\boldsymbol{k}',\boldsymbol{k};E)=\frac{(2\pi)^2}{\mu k_r}\sum_{m=-l_0}^{l_0}Y_{l_0m_0}(\hat{\boldsymbol{k}}')Y_{l_0m_0}^*(\hat{\boldsymbol{k}})\,e^{2i\delta_{l_0}}\frac{\Gamma}{E-E_{nl_0}^{(r)}}.$$

On-shell resonant reaction amplitude
$(k_{xA}^2/2\mu_{xA}=E_{xA},\ k_f^2/2\mu_{bB}=E_f)$



On-shell resonant reaction amplitude (extension of scattering amplitude)

$$M^{(r)}(\boldsymbol{k}_f,\boldsymbol{k}_{xA};E_{xA})=(2\pi)^2\sqrt{\frac{1}{\mu_{xA}k_{(xA)}}}\sqrt{\frac{1}{\mu_{bB}k_f}}$$
$$\times\sum_{m_0=l_0}^{l_0}Y_{l_0m_0}(\hat{\boldsymbol{k}}_f)Y_{l_0m_0}^*(\hat{\boldsymbol{k}}_{xA})\,e^{i(\delta_{(xA)l_0}(k_{(xA)r})+\delta_{fl_0}(k_f))}\frac{\sqrt{\Gamma_{xA}\Gamma_{bB}}}{E_{xA}-E_{(xA)}^{(r)}}$$

Generalization of the resonant reaction amplitude

$$M^{(r)}(\boldsymbol{k}_f,\boldsymbol{k}_{xA};E_{xA})=(2\pi)^2\sqrt{\frac{1}{\mu_{xA}k_{(xA)}}}\sqrt{\frac{1}{\mu_{bB}k_f}}$$
$$\times e^{i(\delta_{(xA)l_0}(k_{xA})+\delta_{fl_0}(k_f))}\sum_{m_0=l_0}^{l_0}Y_{l_0m_0}(\hat{\boldsymbol{k}}_f)Y_{l_0m_0}^*(\hat{\boldsymbol{k}}_{xA})$$
$$\times\frac{\sqrt{\Gamma_{xA}(E_{xA},r_0)\Gamma_{bB}(E_{bB},r_0)}}{E_{xA}-E_{(xA)}^{(r)}}$$

On-shell resonant reaction total cross section

$$\sigma(E_{xA})=\frac{\mu_{xA}k_{xA}\,\mu_{bB}k_f}{(2\pi)^2}\frac{1}{4\pi}\int d\Omega_{\boldsymbol{k}_f}\int d\Omega_{\boldsymbol{k}_{xA}}\,|M^{(r)}(\boldsymbol{k}_f,\boldsymbol{k}_{xA};E_{xA})|^2$$
$$=\frac{(2\pi)^4}{\mu_{xA}k_{xA}}\frac{1}{\mu_{bB}k_f}\frac{\mu_{xA}k_{xA}\,\mu_{bB}k_f}{(2\pi)^2}\frac{1}{k_{xA}^2}\frac{\Gamma_{xA}(E_{xA},r_0)\Gamma_{bB}(E_{bB},r_0)}{(E_{xA}-E_{xA}^{(0)})^2+\frac{\Gamma^2}{4}}$$
$$\times\frac{1}{4\pi}\int d\Omega_{\boldsymbol{k}_f}\int d\Omega_{\boldsymbol{k}_{xA}}\Big[\sum_{m_0=-l_0}^{l_0}Y_{l_0m_0}(\hat{\boldsymbol{k}}_f)Y_{l_0m_0}^*(\hat{\boldsymbol{k}}_{xA})\Big]$$
$$\Big[\sum_{m_0'=-l_0}^{l_0}Y_{l_0m_0'}^*(\hat{\boldsymbol{k}}_f)Y_{l_0m_0'}(\hat{\boldsymbol{k}}_{xA})\Big]$$
$$=(2l_0+1)\frac{\pi}{k_{xA}^2}\frac{\Gamma_{xA}(E_{xA},r_0)\Gamma_{bB}(E_{bB},r_0)}{(E_{xA}-E_{xA}^{(0)})^2+\frac{\Gamma^2}{4}}$$

Half-Off-Shell Scattering Amplitude
(ingoing state is off-shell, final state is on-shell) $(k'^2/2\mu=E,\ k^2/2\mu\neq E)$

$$T^{(r)}(\boldsymbol{k}',\boldsymbol{k};E)=4\pi\sum_{l'=0}^{\infty}i^{l'}(-1)^{l'}e^{i\sigma_{l'}}\sum_{m'=-l'}^{l'}Y_{l'm'}(\hat{\boldsymbol{k}}')4\pi\sum_{l=0}^{\infty}i^{l}\sum_{m=-l}^{l}Y_{lm}^*(\hat{\boldsymbol{k}})$$
$$\times\sum_{m=-l_0}^{l_0}\Big[\int d\boldsymbol{r}'\,\psi_{l'k}^C(r')Y_{l'm'}^*(\hat{\boldsymbol{r}}')V^N(r')\psi_{nl_0}^{(r)}(r')Y_{l_0m_0}(\hat{\boldsymbol{r}}')$$
$$\times\int d\boldsymbol{r}\,\psi_{nl_0}^{(r)}(r)Y_{l_0m_0}^*(\hat{\boldsymbol{r}})V^N(r)\,j_l(kr)Y_{lm}(\hat{\boldsymbol{r}})\Big]\frac{1}{E-E_{nl_0}^{(r)}}$$
$$=(4\pi)^2\sum_{m=-l_0}^{l_0}Y_{l_0m_0}(\hat{\boldsymbol{k}}')Y_{l_0m_0}^*(\hat{\boldsymbol{k}})e^{i\sigma_{l_0}}\frac{W_{l_0}'(k',k_r)W_{l_0}(k,k_r)}{E-E_{nl_0}^{(r)}}$$

There is no Coulomb barrier for virtual particles.

Off-shell resonant form factor

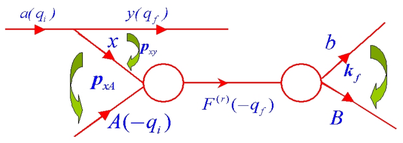
$$w_{l_0}(k,k_r)=(E_{nl_0}^{(r)}-E_k)\int_0^{\infty}dr\,r^2\psi_{nl_0}^{(r)}(r)\,j_{l_0}(kr)$$
$$=(E_{nl_0}^{(r)}-E_k)\,\psi_{nl_0}^{(r)}(k)$$

Half-off-shell resonant scattering amplitude
(ingoing state is off-shell, final state is on-shell)

$$M^{(r)}(\boldsymbol{k}_f,\boldsymbol{p}_{xA};E_{xA})=-\frac{1}{2}(4\pi)^2\sqrt{\frac{1}{\mu_{bB}k_f}}\sum_{m_0=l_0}^{l_0}Y_{l_0m_0}(\hat{\boldsymbol{k}}_f)$$
$$\times Y_{l_0m_0}^*(\hat{\boldsymbol{p}}_{xA})\,e^{i\delta_{fl_0}(k_f)}\frac{\sqrt{\Gamma_{bB}(E_f,r_0)}\,w_{l_0}(p_{xA},k_{(xA)r})}{E_{xA}-E_{xA}^{(r)}}$$

Trojan Horse

Half-Off-Shell Scattering Amplitude
(ingoing state is off-shell, final state is on-shell)
$(p_{xA}^2/2\mu_{xA}\neq E_{xA},\ k_f^2/2\mu_{bB}=E_f)$



Trojan Horse Half-off-shell resonant reaction amplitude

$$M_{TH}(\boldsymbol{q}_f,\boldsymbol{p}_{xA},\boldsymbol{q}_i)=M_{TH}^{(r)}(\boldsymbol{k}_f,\boldsymbol{p}_{xA};E)\varphi_{xy}(\boldsymbol{p}_{xy})$$
$$M_{TH}^{(r)}(\boldsymbol{k}_f,\boldsymbol{p}_{xA};E_{xA})=-\frac{1}{2}(4\pi)^2\sqrt{\frac{1}{\mu_{bB}k_f}}\sum_{m_0=l_0}^{l_0}Y_{l_0m_0}(\hat{\boldsymbol{k}}_f)Y_{l_0m_0}^*(\hat{\boldsymbol{p}}_{xA})$$
$$\times e^{i\delta_{fl_0}(k_f)}\frac{\sqrt{\Gamma_{bB}(E_f,r_0)}\,w_{l_0}(p_{xA},k_{(xA)r})}{E_{xA}-E_{xA}^{(r)}}$$

Virtuality factor

$$\sigma_x=E_x-\frac{p_x^2}{2m_x}=E_{xA}-\frac{p_{xA}^2}{2\mu_{xA}}=-\frac{1}{2\mu_{xy}}[p_{xy}^2+(\kappa_{xy}^a)^2]<0$$
$$(\kappa_{xy}^a)^2=2\mu_{xy}\varepsilon_{xy}^a$$

In Trojan Horse amplitude always $E_{xA}<\frac{p_{xA}^2}{2\mu_{xA}}$

Relationship between the on-shell resonance amplitude and Trojan Horse

$$M^{(r)}(\boldsymbol{k}_f,\boldsymbol{k}_{xA};E)=-\frac{1}{2}\sqrt{\frac{1}{\mu_{xA}k_{xA}}}e^{i\delta_{(xA)l_0}(k_{xA})}\frac{\sqrt{\Gamma_{xA}(E_{xA},r_0)}}{w_{l_0}(p_{xA},k_{(xA)r})}M_{TH}^{(r)}(\boldsymbol{k}_f,\boldsymbol{p}_{xA};E)$$

On-shell resonant reaction cross section expressed in terms of the half-off-shell Trojan Horse amplitude

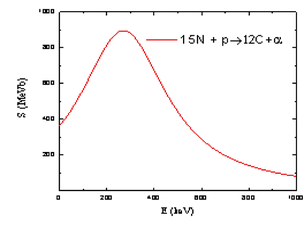
$$\sigma(E_{xA})=\frac{\mu_{xA}k_{xA}\,\mu_{bB}k_f}{(2\pi)^2}\frac{1}{4\pi}\int d\Omega_{\boldsymbol{k}_f}\int d\Omega_{\boldsymbol{k}_{xA}}\,|M^{(r)}(\boldsymbol{k}_f,\boldsymbol{k}_{xA};E_{xA})|^2$$
$$=\frac{1}{4}\frac{\mu_{bB}k_f}{(2\pi)^2}\frac{1}{4\pi}\frac{\Gamma_{xA}(E_{xA},r_0)}{|w_{l_0}(p_{xA},k_{(xA)r})|^2}$$
$$\times\int d\Omega_{\boldsymbol{k}_f}\int d\Omega_{\boldsymbol{k}_{xA}}\,|M_{TH}^{(r)}(\boldsymbol{k}_f,\boldsymbol{p}_{xA};E_{xA})|^2$$

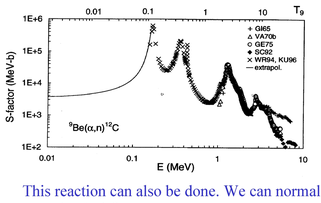
Astrophysical factor:

$$S(E_{xA})=E_{xA}e^{2\pi\eta_{xA}}\sigma(E_{xA})$$
$$=E_{xA}e^{2\pi\eta_{xA}}\frac{1}{4}\frac{\mu_{bB}k_f}{(2\pi)^2}\frac{1}{4\pi}\frac{1}{k_{xA}^2}\frac{\Gamma_{xA}(E_{xA},r_0)}{|w_{l_0}(p_{xA},k_{(xA)r})|^2}$$
$$\times\int d\Omega_{\boldsymbol{k}_f}\int d\Omega_{\boldsymbol{k}_{xA}}\,|M_{TH}^{(r)}(\boldsymbol{k}_f,\boldsymbol{p}_{xA};E_{xA})|^2$$
$$=E_{xA}e^{2\pi\eta_{xA}}\frac{P_{l_0}(E_{xA},r_0)}{P_{l_0}(E_{xA}^{(0)},r_0)}\frac{1}{4}\frac{\mu_{bB}k_f}{(2\pi)^2}\frac{1}{4\pi}\frac{1}{2\mu_{xA}E_{xA}}\frac{\Gamma(E_{xA}^{(0)})}{|w_{l_0}(p_{xA},k_{(xA)r})|^2}$$
$$\times\int d\Omega_{\boldsymbol{k}_f}\int d\Omega_{\boldsymbol{k}_{xA}}\,|M_{TH}^{(r)}(\boldsymbol{k}_f,\boldsymbol{p}_{xA};E_{xA})|^2$$
$$\sim_{E_{xA}\to 0}\frac{\Gamma(E_{xA}^{(0)})}{P_{l_0}(E_{xA}^{(0)},r_0)}\frac{1}{4}\frac{1}{2\mu_{xA}}\frac{\mu_{bB}k_f}{(2\pi)^2}\frac{1}{4\pi}\frac{r_0}{|w_{l_0}(p_{xA},k_{(xA)r})|^2}$$
$$\times\int d\Omega_{\boldsymbol{k}_f}\int d\Omega_{\boldsymbol{k}_{xA}}\,|M_{TH}^{(r)}(\boldsymbol{k}_f,\boldsymbol{p}_{xA};E_{xA})|^2$$

Example; $^{15}N+p\to{}^{16}O(12.44\,\text{MeV})\to{}^{12}C+\alpha.$

This reaction is supposed to be done here in collaboration with Catania National Lab.





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This reaction can also be done. We can normalize the astrophysical factor.

$$S(E_{xA})=E_{xA}e^{2\pi\eta_{xA}}\frac{P_{l_0}(E_{xA},r_0)}{P_{l_0}(E_{xA}^{(0)},r_0)}\frac{1}{4}\frac{\mu_{bB}k_f}{(2\pi)^2}\frac{1}{4\pi}\frac{1}{2\mu_{xA}E_{xA}}\frac{\Gamma(E_{xA}^{(0)})}{|w_{l_0}(p_{xA},k_{(xA)r})|^2}$$
$$\times\int d\Omega_{\boldsymbol{k}_f}\int d\Omega_{\boldsymbol{k}_{xA}}\,|M_{TH}^{(r)}(\boldsymbol{k}_f,\boldsymbol{p}_{xA};E_{xA})|^2$$

There is no Coulomb barrier for virtual particles.

Here the transferred proton is virtual, and there is no Coulomb barrier between the proton and deuteron. The astrophysical factors for both reactions at $E\to 0$ practically coincide.

Summary

1. A comparative analysis of the resonant on-shell and half-off-shell amplitudes has been done focusing on the impact of the off-shell effects.
2. It has been shown that there is no Coulomb barrier for off-shell particles. It explains why the Trojan Horse method can be used to measure resonant reactions at astrophysically relevant energies.
3. A relationship connecting the resonant Trojan Horse and on-shell amplitudes is presented.
4. The resonant cross section and the astrophysical factor are expressed in terms of the Trojan Horse amplitude.
5. The absence of the Coulomb barrier for off-shell particles explains why the astrophysical factors for reactions $d+d\to n+{}^3He$ and $d+d\to p+{}^3H$ practically are the same.
6. In the future, the contribution of the background should be taken into account. The background can interfere with the resonant amplitude. In addition, the scattering of the particles in the initial and final states of the Trojan Horse reaction should be taken into account.