Coherent Polychotomous Waves From An Attractive Well

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This work deals with the classical textbook exercise of a wave packet interacting with an attractive well [1]. Despite being a thoroughly studied example of quantum scattering for plane wave stationary states, the effect to be presented here for wave packets was yet to be found.

A one-dimensional attractive well can either reflect or transmit a wave. Reflection and transmission coefficients are the simplest scattering amplitudes. They can easily be calculated for a square well by using plane waves and elementary continuity conditions. The analysis of the exact time development of a packet, as well as the treatment of realistic well shapes, is however reserved for numerical treatment.

We here show that wave packet scattering possesses an intriguing aspect: Packets that are narrower than the well width initially, resonate inside it, generating a reflected wave that is coherent and monochromatic in amplitude, a polychotomous wave train.

Polychotomous (multipeak) waves are observed when a superintense laser field focuses on an atom [2]. Ionization is hindered and the wave function is localized, in spite of the presence of the strong radiation field. The wave packet representing the excited electron eventually spreads and the degree of localization and/or ionization depends on the parameters of the radiation field. The above effect appears when the external field operates on a bound state.

The effect may be tested in back angle nuclear reactions and an estimate of the energy, projectiles and candidate targets will be given below. The effect is analogous to lasing inside a cavity, the well becomes then the most natural laser available.

Consider a minimal uncertainty wave packet traveling from the left with an average speed v, initial location x_0, mass m, wave number $q = m\, v$ and initial width δ,

$$\psi = C\, exp\Big(i\, q\, (x - x_0) - \frac{(x - x_0)^2}{4\, \delta^2}\Big) \quad (1)$$

impinging on an attractive well located at the origin, with depth A and width w. For the sake of simplicity we use an exponential well, but the results are not specific to the type or shape of the well

$$V(x) = -A\, exp\Big(-\frac{x^2}{w^2}\Big) \quad (2)$$

We solve the Schrödinger equation for the scattering event in coordinate space taking care of unitarity. We use the method of Goldberg et al. [3], that proved to be extremely robust and conserves the wave normalization with an error of less than 0.01 %, even after hundreds of thousands of time step iterations. We have verified that the solutions actually solve the equation with extreme accuracy by explicit substitution. Other simple discretization methods of resolution such as Runge-Kutta, leapfrog, etc., are unstable for this type of equation, they violate unitarity.

We study the scattering of an impinging packet with $\delta = 0.5$ and a well width of $w = 1$. We also use a large mass $m = 20$ in order to prevent the packet from spreading too fast [1].

A polychotomous (multiple peak) wave recedes from the well. For low velocities, corresponding to average packet energies less than half the well depth, several peaks in the reflected wave show up. Simple inspection reveals that the distance between the peaks is constant. The reflected wave is propagating with an amplitude of the form

$$C(x) \approx e^{-\lambda|x|}\, sin^2(kx) \tag{3}$$

The exponential drop is characteristic of a bound state solution inside the well. The parameters λ ad k, are independent of the initial velocity, but depend on time. The wave spreads and its amplitude diminishes, as expected. We have checked that the polychotomous behavior continues for $t \rightarrow \infty$ without modification.

The transmitted packet travels at a different velocity than that the coherent reflected packet. The reflected packet travels with a constant speed of $v = k(t_{formation})/m$. The speed can be found by evaluating the effective center of mass X

$$X_{refl} = \int_{-\infty}^{-w} dx\ x\ |\psi(x)|^2 \tag{4}$$

Where the wave function is properly normalized to 1. Using the above equation one finds that the reflected wave center of mass recedes with a constant speed of approximately $v = 0.03$ independent of the initial speed of the incident packet, while the transmitted wave rides away with a velocity slightly higher than the initial packet average velocity, and it is determined by overall energy conservation.

The polychotomous effect disappears when the wave packet is broader than the well.

We have also investigated other types of wells, such as a Lorentzian well, a square well, etc., and found the same phenomena described here. Moreover the effect is independent of the shape of the packet as long as it is narrower than the well width. We have used square packets, Lorentzian packets, linear exponential packets, etc., with analogous results.

Let us consider now the conditions for the effect to be measured experimentally. Consider for example backward angle scattering of neutrons, or protons on nuclei. Although our treatment was one-dimensional, it should apply also for the case of zero angular momentum in three dimensions. Typical nuclear well depths are around 30 MeV, with widths of around a few Fermi for light nuclei, hence $k' \approx 1.25 fm^{-1}$. The condition for the excitation of the metastable resonance in the well and the coherence of the reflected wave can be met easily. For a nucleon of 20 MeV energy we find that a 30 MeV well satisfies the condition if it exceeds 3 Fermi in radius. A nucleus like O^{16} may very well serve for that purpose. The kinetic energy of 20 MeV is above the Coulomb barrier for light nuclei, hence we do not expect major distortions in the reflected wave when protons are used. Conversely, the effect may serve as a method to determine nuclear well depths (or radii) by merely registering the dead time between bunches in the reflected wave beam. The effect may also be tested in atomic collisions at backward angles.

References

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