Decay To Bound States Of A Soliton In A Well

G. Kälbermann

Topologically stable solitons arise in field theories with nonlinear self-interactions. When a soliton scatters off an attractive impurity it may be trapped [1, 2](see also ref. [3, 4]). This behavior can be understood in terms of a few degrees of freedom for the soliton [1]. We here address the decay of the soliton from a trapped state to a bound state.

Consider the kink lagrangian

$$\mathcal{L} = \frac{\partial_\mu \phi}{\partial^\mu \phi} - \frac{1}{4}\Lambda \left(\phi^2 - \frac{m^2}{\lambda}\right)^2 \tag{1}$$

Here

$$\Lambda = \lambda + V(x) \tag{2}$$

λ being a constant, and $V(x)$ the impurity potential [2]

$$V(x) = h\ cosh^{-2}\Big(a\ (x - x_c)\Big) \tag{3}$$

Independently of the choice of parameters it is found that trapped states decay. When the soliton reaches the well, it oscillates and starts to emit radiation. The emission of radiation damps the oscillations. After a certain time, and due to the finite extent of the x-axis, radiation reflects back from the boundaries and reaches the soliton. The soliton subsequently absorbs the radiation and its amplitude starts to increase. The time taken for radiation to return to the soliton is the travel time for the fastest 'mesons' of the theory.

The dispersion relation for the radiated mesons can be extracted from the expansion of the scalar field around the soliton solution. Using $\lambda = m = 1$ we find $\omega^2 = k^2 + 2$. The velocity of the mesons is bounded by

$$u_{max} = \frac{\omega}{k}\Big)_{max} = 1.$$

The frequency of the oscillations of the soliton in a trapped state may be estimated analytically. Using an expansion of the potential in eq. (3) around the bottom of the well $V(x) \approx -V_0 + \epsilon\ y^2$, $y = x - x_c$ and an ansatz appropriate for small oscillations of the soliton around the center of the well $\phi \approx (y + \delta\ y^3/2)\ sin(\omega(t - t_0))$ we find $\omega^2 = 2\ \mu$.

With $\mu \approx \pm\sqrt{\frac{4}{5}\epsilon + \frac{9}{10}(V_0 - 1)}$. (The positive solution has to be chosen)

The formula compares reasonably well with the leading frequency of oscillation of the soliton inferred from a Fourier analysis of the amplitude of the field at the center of the well. However, the fluctuation of the soliton in the well is anharmonic.

After emission of radiation the soliton becomes bound in the well. These bound states exist not only for a soliton centered with the well, but also for a soliton located off-center. The former are produced after the soliton radiates its kinetic energy, while the latter seem more difficult to realize. Perhaps multisoliton collisons may lead to a degradation of the kinetic energy of one such soliton off-center thereby trapping it at rest in a location at which it would be unstable as a classical particle. This phenomenon has no counterpart in the classical behavior of particles. Only the bottom of the well is a stable point for the particles. Again, it seems that the extended character of the soliton is playing a major role in generating such unexpected solutions.

It is clear that these off-center solutions are true bound states, because their energy is smaller than the free soliton mass. However, any small perturbation of the soliton will make it drift to the center of the well. The off-center solutions are in this sense unstable.

We found the bound state solutions, by integrating the static equations of motion starting from the center of the soliton. There appears to be only a single bound state for each choice of well depth and width, even for large well depths.

References

[1] Y. S. Kivshar, Z. Fei and L. Vazquez, Phys. Rev. Lett. **67**, 1177 (1991), Z. Fei, Y. S. Kivshar and L. Vazquez, Phys. Rev. **A46**, 5214 (1992).

[2] G. Kälbermann, Phys. Rev. **E55**, R6360 (1997).

[3] J. A. Gonzalez and B. de A. Mello, Phys. Scripta **54**, 14 (1996),

[4] J. A. Gonzalez, B. de A. Mello, L. I. Reyes and L. E. Guerrero, Phys. Rev. Lett. **80** (1998) 1361.