3.4 Instantons and Light Quarks

$$Z = \int \mathcal{D}A_\mu \, \mathcal{D}\Psi^+ \mathcal{D}\Psi \; e^{-S^E_{glue} - S^E_{quark}}$$

quark action $\quad S^E_{quark} = -\int d^4x \, \bar{\Psi} \, (i \not{D} + i m_q) \Psi$

$$= \frac{1}{N_+! \, N_-!} \prod_{I, \bar{I}}^{N_+, N_-} \int d\Omega_I \, n(\rho_I) \, e^{-S_{int}} \prod_f^{N_f} \det(i \not{D} + i m_f)$$

contains instanton field

eigenfunctions /-values $\quad i \not{D} \, \Psi_\lambda = \lambda \, \Psi_\lambda$

$\Rightarrow \not{D}$ diagonal $\Rightarrow \det(i \not{D} + i m_f) = \prod_\lambda (\lambda + i m_f)$ product of eigenvalues

problem: in the field of a single (anti-) instanton,

$\exists$ one left- (right-) handed quark "zero-mode" $\Psi_{\lambda=0}$ (per flavour $u, d, \ldots$)

$\Rightarrow$ tunneling amplitude $dn_I \propto m_f^{N_f} \to 0$ for $m_f \to 0$

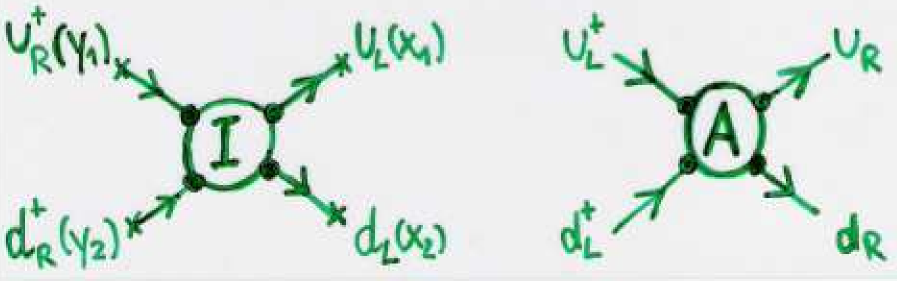
instantons suppressed by massless quarks ?!

remedy: quark propagation $\quad (i \not{D} + i m_q) \, G_q(x,y) = \delta^{(4)}(x-y) \, 1$

$$\Rightarrow G_q(x,y) = \frac{\Psi_0(x) \, \Psi_0^+(y)}{i m_q} + \sum_{\lambda \neq 0} \frac{\Psi_\lambda(x) \, \Psi_\lambda^+(y)}{\lambda + i m_q}$$

in the presence of an instanton:

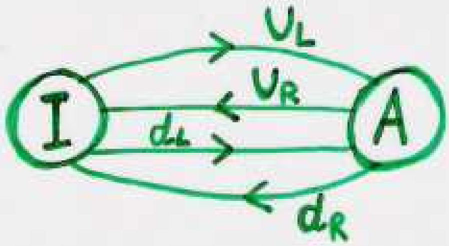
$\langle n_w = 1 | \prod_f \Psi_f(x) \Psi_f^+(y) | n_w = 0 \rangle \propto (m_q)^{N_f} \frac{1}{m_q^{N_f}}$ finite, dominated by zero-mode propagator



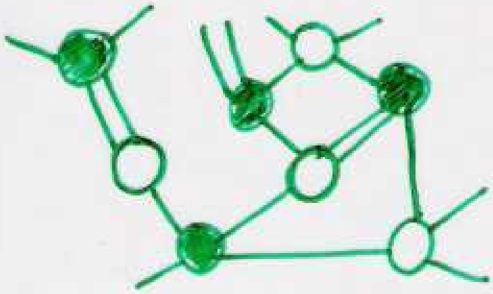
Where do the quarks go?

i.e. How to close lines?

I-A "molecules"



I-A "network"



more formally

$$\mathcal{Z}_{inst} = \frac{1}{N_+!\,N_-!} \prod_{I,\bar{I}}^{N_+,N_-} \int d\Omega_I \, n(\rho_I) \, e^{-S_{int}} \prod_f^{N_f} \det\left[i\not{D}(A^\mu_I, A^\mu_{\bar{I}}) + im \right.$$

in zero-mode subspace :

$$[i\not{D} + im_f] = \begin{array}{cc} \begin{array}{cc} N_+ & N_- \end{array} & \\ \left[\begin{array}{c|c} im_f \hat{1} & \hat{T}_{IA} \\ \hline \hat{T}_{AI} & im_f \hat{1} \end{array}\right] & \begin{array}{c} N_+ \\ N_- \end{array} \end{array}$$

Weyl basis

$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$

$\gamma_5 = \begin{pmatrix} 1 & \\ 0 & - \end{pmatrix}$

$\gamma_0 = \begin{pmatrix} 0 & \\ 1 & \end{pmatrix}$

$$\boxed{\begin{aligned} T_{IA}(\Omega_I - \Omega_A) &= \langle \Psi^+_{0I} | i\not{D} | \Psi_{0,A} \rangle \\ &= \int d^4x \, \Psi^+_{0I,R}(x - z_I) \, i\not{D}_x \, \Psi_{0A,R}(x - z_A) \end{aligned}} = T^+_{AI}$$

Interpretation: $T_{IA} = \int d^4x \, d^4y \, (\Psi^+_{0I} \not{D}_y) \underbrace{\not{D}_y^{-1} \delta^{(4)}(x-y)}_{\equiv G_q(x,y)} (i\not{D}_x \Psi_{0A})$

amplitude for instanton to emit quark

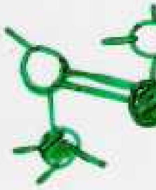
$\hat{=}$ I-A interaction via quark exchange !

⇒ 2-component partition function (MFA):

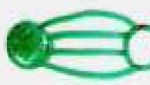
$$\mathcal{Z}^{atm+mol}_{grand} = \sum_{N_a, N_m} \frac{(z_a V_4)^{N_a}}{N_a!} \frac{(z_m V_4)^{N_m}}{N_m!}$$

$$\Rightarrow \boxed{\Omega^{a+m}_{can.}} = -\frac{1}{V_4} \log(\mathcal{Z}^{a+m}_{can}) = \boxed{-n_a\left[\log\left(\frac{z_a}{n_a}\right)+1\right] - n_m\left[\log\left(\frac{z_m}{n_m}\right)+1\right]}$$

"atomic" component: $z_a = 2\,\mathrm{const}\; e^{-\frac{1}{2}S_{int}} \left(\langle T_{IA} T_{AI}\rangle \frac{n_a}{2}\right)^{\frac{N_f}{2}}$ "random" I-A



"molecular" component: $z_m = \mathrm{const}^2\; e^{-S_{int}} \langle (T_{IA} T_{AI})^{N_f}\rangle$ "bound" I-A



→ minimize $\Omega^{a+m}_{can.}$ (fix $\frac{N}{V_4} = n_a + 2n_m \overset{!}{=} 1.4\,\mathrm{fm}^{-4} \Leftrightarrow \Lambda_{QCD} \simeq 260\,\mathrm{MeV}$)

find: $n_a = 1.34\,\mathrm{fm}^{-4}$, $n_m = 0.03\,\mathrm{fm}^{-4}$ $\hat{=}$ "random" I-A vacuum,
quark-zero modes <u>de</u>-localized

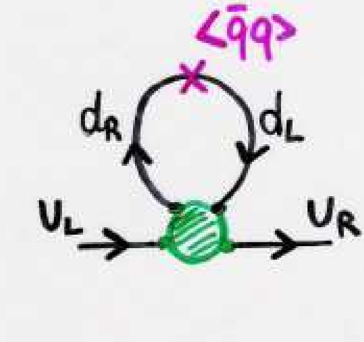
⇒ Spontaneous Breaking of Chiral Symmetry (SBCS)

effective quark mass from determinant:

$$\boxed{M^*_q} = m_q + \left(\langle |T_{IA}|^2\rangle \frac{n_a}{2}\right)^{1/2}$$

$$= \boxed{m_q - \lambda \langle \bar{q}q\rangle}$$

quark selfenergy in "instanton <u>liquid</u>"



$$\boxed{\langle\bar{q}q\rangle} \equiv \mathrm{Tr}\, G_q(x,x)$$

$$= -\int \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\, G_q(p) = \boxed{-4N_c \int \frac{d^4p}{(2\pi)^4} \frac{M^*_q}{p^2 + M^{*2}_q}}$$

$$\text{"}\ \not{p} + M^*_q / (p^2 + M^{*2}_q)$$

selfconsistent ("gap") equation

alternatively

integrate-out gluonic fields to obtain effective (anti-) fermionic action

$\Rightarrow$ (anti-) instantons provide $2N_f$-quark interactions:

$$\mathcal{L}_{eff}^{inst} = \tilde{\lambda}\ \left[\varepsilon_{ff'} \varepsilon_{gg'} (\bar{q}_R^f \mathcal{F}^+ q_L^g \mathcal{F})(\bar{q}_R^{f'} \mathcal{F}^+ q_L^{g'}) + -\bar{I}- \right]$$

$$\equiv \lambda\ \left[(\bar{q}\, \underset{\uparrow}{\mathcal{F}^+}\, \vec{\tau}\, \underset{\uparrow}{\mathcal{F}} q)^2 + (\bar{q}\, \underset{\uparrow}{\mathcal{F}^+}\, \vec{\tau}\, \gamma_5\, \underset{\uparrow}{\mathcal{F}} q)^2 \right]$$

form factors $\hat{=}$ instanton zero-modes

"range" $\sim \frac{1}{\rho} \simeq 600\,\mathrm{MeV}$

attractive for "σ" $= \bar{q}q$ scalar-isoscalar

"$\vec{\pi}$" $= \bar{q}\vec{\tau}\gamma_5 q$ pseudoscalar-isovector

repulsive for "$\vec{a}_0$" $= \bar{q}\vec{\tau}q$ scalar-isovector

"η'" $= \bar{q}\gamma_5 q$ pseudoscalar-isoscalar

no axial-/vector interaction at 1-instanton level ("$\vec{a}_1$") ("$\vec{\rho}$")

4.) Finite-T χiral Phase Transition + Instanton

→ impose (anti-) periodic boundary conditions on (quark-) gluon-fields ⇒ explicit finite-T solutions for:

- instanton gauge fields $A^a_\mu(r,\tau;\mathbf{T})$
- quark zero modes $\Psi_{0,I}(r,\tau;\mathbf{T}) \propto e^{-\pi r T}$ (enhanced in τ)

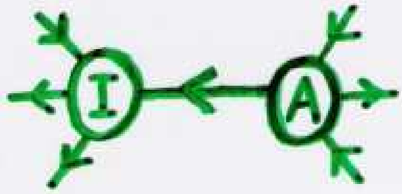
QCD partition function

$$Z^{inst}_{QCD} = \sum_{N_+,N_-} \frac{1}{N_+!\,N_-!} \prod_{I=1}^{N_+,N_-} \int_{\tau=0}^{\beta} \int d\Omega_I \, n(\rho_I)\, e^{-S_{int}} \rho_I^{N_f} \, |T_{IA}(z,u;\mathbf{T})|^2$$

Quark-induced Instanton-Antiinstanton interaction:

$$T_{IA}(z,u;\mathbf{T}) = \int_0^\beta dx_4 \int d^3x \, \Psi^+_{0,I}(x-z_I;\mathbf{T}) \, i\not{D}(\mathbf{T}) \, \Psi_{0,A}(x-z_A;\mathbf{T})$$

$$\equiv i u_4 \, f_1(r,\tau;T) + i \frac{\vec{u}\cdot\vec{r}}{r} f_2(r,\tau;T)$$



Chiral Symmetry Restoration

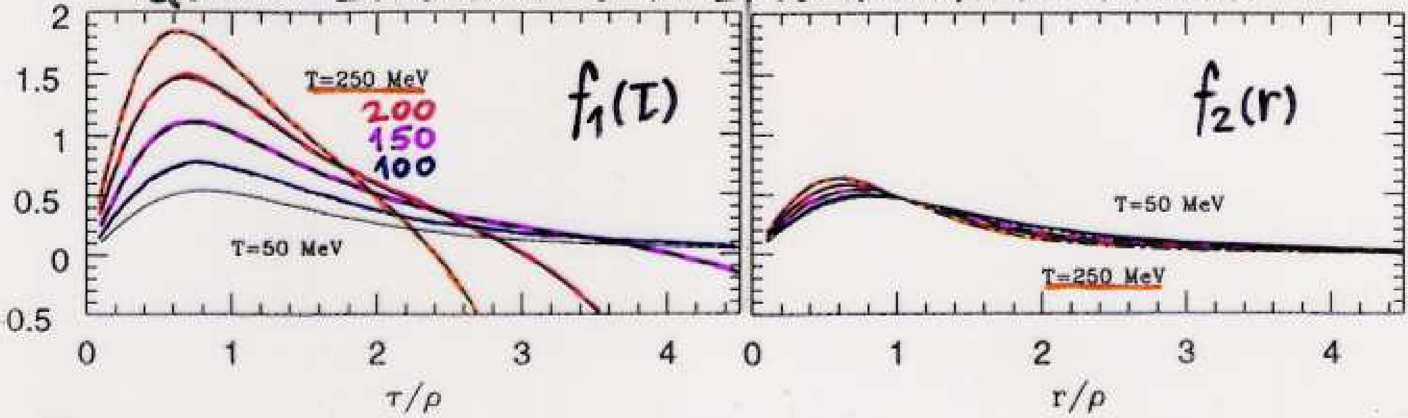
naively: Debye-screening of **I**-/**A**-fields

but: sets in only above T_c (Chu+Schramm, PRD '95)

⇒ other mechanism operative

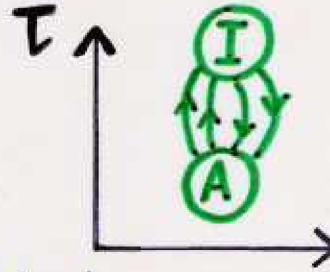
observation: T_{IA} strongly **T**-dependent: suppressed in **r**-direction
enhanced " τ- "

Quark-Induced I-A-Interaction at Finite T



⇒ Formation of Instanton-Antiinstanton 'Molecules'

≡ **I-A** pairs localized in space and aligned in time direction



Schematic **'Cocktail'** Model for Thermodynamic Potential:

(Ilgenfritz+Shuryak, PLB '94)

$$\Omega^{inst}(T) = -\frac{\log(Z^{inst}(T))}{V_4} \simeq -n_a\left(1+\log\left(\frac{z_a}{n_a}\right)\right) - n_m\left(1+\log\left(\frac{z_m}{n_m}\right)\right)$$

'atomic activity' $\quad z_a = 2C\rho^{b-4}e^{-S_{int}}\left(\langle|T_{IA}|^2\rangle\frac{n_a}{2}\right)^{N_f/2}$

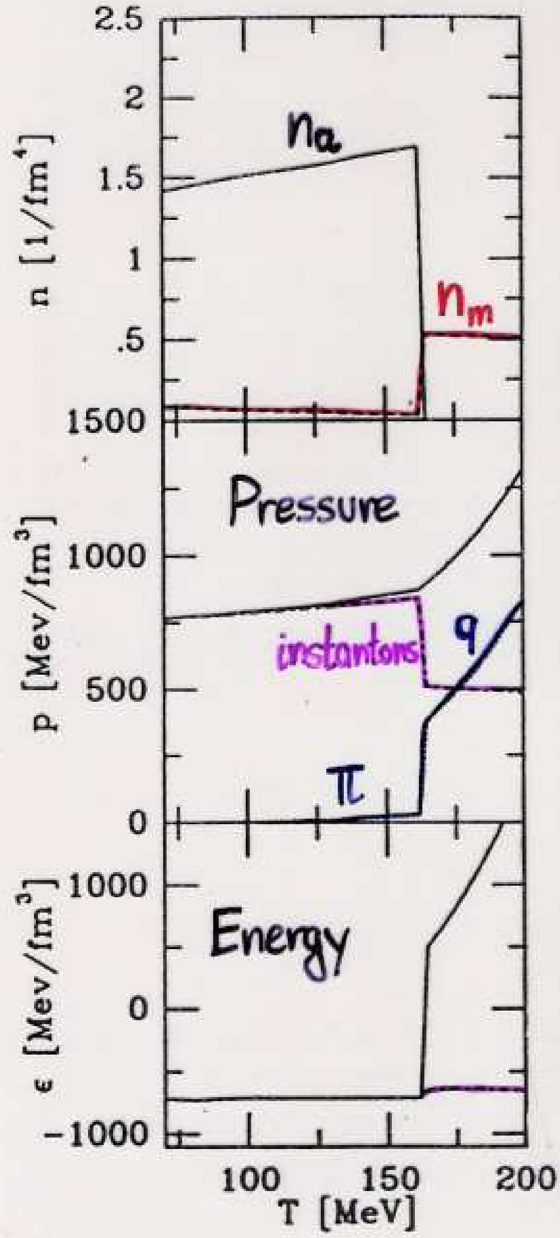
'molecular activity' $\quad z_m = C^2\rho^{2(b-4)}e^{-2S_{int}}\langle|T_{IA}|^{2N_f}\rangle$

(gluonic interaction S_{int} weakly temperature-dependent [Diakonov+Mirlin PLB '88])

⇒ $\Omega^{inst}(T)$ minimized w.r.t. concentrations n_a, n_m

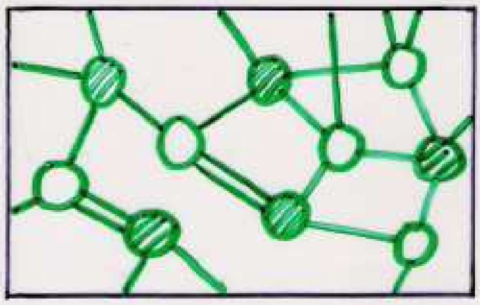
(normalization $C \propto \Lambda_{QCD}^b$ fixed at $T=0$ to give $n(T=0)=1.4\,\mathrm{fm}^{-4}$)

Chiral Symmetry Restoration at $T>0$ $(\mu_q=0)$



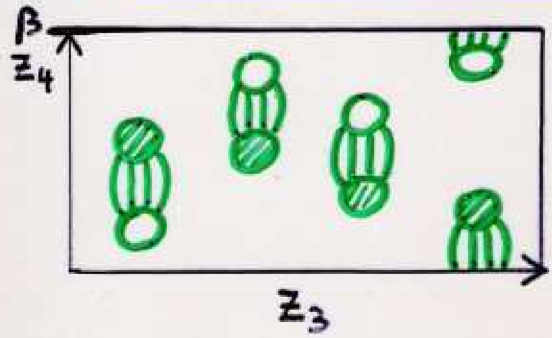
due to **I-A**-molecule formation:

$T=0$: random liquid



$\langle\bar{q}q\rangle \propto \sqrt{n_a}$ finite

$T>T_c^\chi \simeq 150\,\text{MeV}$: molecular phase



$\langle\bar{q}q\rangle \propto \sqrt{n_a} = 0$

(Ilgenfritz+Shuryak PLB '94)

also found in full numerical simulations of the **IILM**

(Schäfer, Shuryak, Verbaarschot PRD '95, '96)