Transformation rules for $\vec{E}$ and $\vec{B}$?

– Transform $F^{\mu\nu}$!

$$F^{\mu\nu'} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F^{\rho\sigma} = \Lambda F \Lambda^{t}$$ as matrix multiplication

$$\Lambda F = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \beta\gamma\frac{E_x}{c} & \gamma\frac{E_x}{c} & \gamma\frac{E_y}{c} - \beta\gamma B_z & \gamma\frac{E_z}{c} + \beta\gamma B_y \\ -\gamma\frac{E_x}{c} & -\beta\gamma\frac{E_x}{c} & -\beta\gamma\frac{E_y}{c} + \gamma B_z & -\beta\gamma\frac{E_z}{c} - \gamma B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$\Rightarrow \quad \Lambda F \Lambda^{t} = \begin{pmatrix} 0 & -\beta^2\gamma^2\frac{E_x}{c} + \gamma^2\frac{E_x}{c} & \gamma\frac{E_y}{c} - \beta\gamma B_z & \gamma\frac{E_z}{c} + \beta\gamma B_y \\ & 0 & -\beta\gamma\frac{E_y}{c} + \gamma B_z & -\beta\gamma\frac{E_z}{c} - \gamma B_y \\ & & 0 & B_x \\ & & & 0 \end{pmatrix}$$

By Anti-Sym

$$\Rightarrow F' = \begin{pmatrix} 0 & \frac{E_x}{c} & \gamma\left(\frac{E_y}{c} - \beta B_z\right) & \gamma\left(\frac{E_z}{c} + \beta B_y\right) \\ -\frac{E_x}{c} & 0 & \gamma\left(B_z - \beta\frac{E_y}{c}\right) & -\gamma\left(B_y + \beta\frac{E_z}{c}\right) \\ -\gamma\left(\frac{E_y}{c} - \beta B_z\right) & -\gamma\left(B_z - \beta\frac{E_y}{c}\right) & 0 & B_x \\ -\gamma\left(\frac{E_z}{c} + \beta B_y\right) & \gamma\left(B_y + \beta\frac{E_z}{c}\right) & -B_x & 0 \end{pmatrix}$$

$$\Rightarrow \quad \frac{E_x'}{c} = \frac{E_x}{c} \qquad \frac{E_y'}{c} = \gamma\left(\frac{E_y}{c} - \beta B_z\right) \qquad \frac{E_z'}{c} = \gamma\left(\frac{E_z}{c} + \beta B_y\right)$$

$$B_x' = B_x \qquad B_y' = \gamma\left(B_y + \beta\frac{E_z}{c}\right) \qquad B_z' = \gamma\left(B_z - \beta\frac{E_y}{c}\right)$$

But these are for $\vec{v} = c\beta\hat{x} \Rightarrow$

$$-\beta B_z = \frac{1}{c}\left(\vec{v}\times\vec{B}\right)_y \qquad \beta B_y = \frac{1}{c}\left(\vec{v}\times\vec{B}\right)_z$$

$$\beta\frac{E_z}{c} = -\left(\frac{\vec{v}}{c}\times\frac{\vec{E}}{c}\right)_y \qquad -\beta\frac{E_y}{c} = -\left(\frac{\vec{v}}{c}\times\frac{\vec{E}}{c}\right)_z$$

$$\Rightarrow \boxed{\begin{array}{ll} \vec{E}_\parallel' = \vec{E}_\parallel & \vec{E}_\perp' = \gamma\left(\vec{E}_\perp + \vec{v}\times\vec{B}\right) \\ \vec{B}_\parallel' = \vec{B}_\parallel & \vec{B}_\perp' = \gamma\left(\vec{B}_\perp - \vec{\beta}\times\frac{\vec{E}}{c}\right) \end{array}}$$

Equivalent to (but more general than) Griffiths Eq. 12.109 pg 559

Can use this, for example, to find $\vec{E}$ and $\vec{B}$ for a point charge moving with a constant velocity $\vec{v}$; starting from the Coulomb field of a point charge at rest ultimately leads to Griffiths Eqs. 10.75 + 10.76.

See Sample Exam, Problem #1

Griffiths goes further, claiming to <u>derive</u> magnetic fields from Coulomb's law and special relativity.

Jackson: <u>Not so fast!</u>

Don't need "everything." But need several assumptions beyond just Coulomb's Law.

Can also use this to identify <u>Lorentz Invariants</u> involving $\vec{E}$ + $\vec{B}$:

Start with $F^{\mu\nu}$ or $G^{\mu\nu}$, combine with "stuff" till all the indices have been contracted away.

If we limit ourselves to functions of $\vec{E}$ + $\vec{B}$, the only options are:

$$F^{\mu\nu}F_{\mu\nu}$$

$$F^{\mu\nu}G_{\mu\nu}$$

$$G^{\mu\nu}G_{\mu\nu}$$

$$F^{\mu\nu}F_{\mu\nu} = 2\left[-\frac{E^2}{c^2} + B^2\right] = \text{Lorentz invariant}$$

Usually simplified to $\boxed{B^2 - \frac{E^2}{c^2} \text{ or } \frac{E^2}{c^2} - B^2}$

Implication: If $B^2 \gtrless \frac{E^2}{c^2}$ in one frame, the same is true in all frames.

$$F^{\mu\nu}G_{\mu\nu} = -4\frac{\vec{E}}{c}\cdot\vec{B} = \text{Lorentz invariant}$$

Usually just quoted as $\boxed{\vec{E}\cdot\vec{B}}$

Implication: If $\vec{E} \perp \vec{B}$ in one frame, then $\vec{E} \perp \vec{B}$ in all frames (or one of the two fields is zero).

$$G^{\mu\nu}G_{\mu\nu} = 2\left[\frac{E^2}{c^2} - B^2\right]$$ doesn't add anything new.

Last Example of PHYS 305: Griffiths Prob. 12.69

Problem 12.69 A charge q is released from rest at the origin, in the presence of a uniform electric field $\mathbf{E} = E_0\,\hat{\mathbf{z}}$ and a uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{x}}$. Determine the trajectory of the particle by transforming to a system in which $\mathbf{E} = \mathbf{0}$, finding the path in that system and then transforming back to the original system. Assume $E_0 < cB_0$. Compare your result with Ex. 5.2.

Consider the fields in S' moving with $\vec{v} = \frac{E_0}{B_0}\hat{y}$ $(<c)$

$$\vec{E}'_{\parallel} = 0 \;; \quad \vec{E}'_{\perp} = \gamma\left[E_0\hat{z} + \left(\frac{E_0}{B_0}\hat{y} \times B_0\hat{x}\right)\right] = 0$$

$$\Rightarrow \underline{\vec{E}' = 0 \text{ in } S'!}$$

$$\vec{B}'_{\parallel} = 0 \;; \quad \vec{B}'_{\perp} = \gamma\left[B_0\hat{x} - \left(\frac{E_0}{B_0 c}\hat{y} \times \frac{E_0}{c}\hat{z}\right)\right]$$

$$= \frac{1}{\sqrt{1 - \left(\frac{E_0}{B_0 c}\right)^2}}\left[\left(B_0 - \frac{E_0^2}{B_0 c^2}\right)\hat{x}\right]$$

Note: $\dfrac{c}{c}\dfrac{B_0 c}{\sqrt{B_0^2 c^2 - E_0^2}}\left(B_0 - \dfrac{E_0^2}{B_0 c^2}\right) = \dfrac{B_0^2 c^2 - E_0^2}{c\sqrt{B_0^2c^2 - E_0^2}} = \sqrt{B_0^2 - \dfrac{E_0^2}{c^2}}$

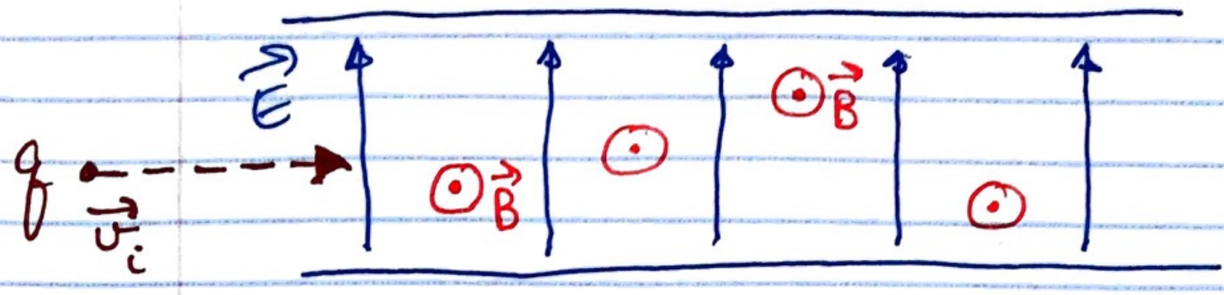
$|\vec{B}'|$ is just what we should have expected since we know $B^2 - \frac{E^2}{c^2}$ is Lorentz invariant!

In S', q will start with velocity $-\frac{E_0}{B_0}\hat{y}$.

Will move in a circular orbit in the y-z plane in S', then S' moves with constant velocity $\frac{E_0}{B_0}\hat{y}$ in S.

Net result is the cycloid motion that was found in Example 5.2 (pg 213) and shown in Figure 5.7 (pg 214).

Alternatively, what if $\vec{v}_i = \frac{E_0}{B_0}\hat{y}$ in S?



In S: $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) = q\left[E_0\hat{z} + \left(\frac{E_0}{B_0}\hat{y} \times B_0\hat{x}\right)\right] = 0$

$\Rightarrow \vec{F}_{tot} = 0$ and particle moves with

$\vec{v} = \text{constant} = \vec{v}_i$

Now view the same particle in S'.

Just sits at rest at the origin

$\Rightarrow$ no magnetic force because $\vec{v} = 0$

$\Rightarrow$ no total force because $\vec{a} = 0$

$$\boxed{\begin{array}{l}\Rightarrow \text{NO ELECTRIC FORCE} \\ \Rightarrow \vec{E}' = 0\end{array}}$$

as we found before!