What about the $\vec{E}$ + $\vec{B}$ fields?

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} \Rightarrow \frac{\vec{E}}{c} = -\frac{\partial}{\partial(ct)}\vec{A} - \vec{\nabla}\left(\frac{V}{c}\right)$$

$$\Rightarrow \frac{E_i}{c} = \partial^0 A^i - \partial^i A^0$$

Not repeating indices ⇒ no Einstein summation here!

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow B_i = \varepsilon_{ijk}\,\partial^j A^k$$

Repeated Roman indices ⇒ two term sums.

But no 4-vector sums that can "remove" indices.

Can combine these together:

Define: $$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu}$$

$$= \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

The "Field Tensor"

$F^{\mu\nu}$ is an anti-symmetric rank-two tensor

⇒ has 6 independent components: $\vec{E}$ + $\vec{B}$!

$\vec{E}$ and $\vec{B}$ are NOT the space parts of two different 4-vectors.

Rather, they appear together as parts of a single common tensor.

$\Rightarrow$ Lorentz transformations will MIX THEM!!

More on that later. First...

$\boxed{\text{What about Maxwell's Equations?}}$

We found $\Box^2 A^\mu = -\mu_0 J^\mu$ with $\partial_\mu A^\mu = 0$

is equivalent to the inhomogeneous Maxwell's Eqs.

What about $\partial_\mu F^{\mu\nu}$?

$$\partial_\mu F^{\mu\nu} = \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$
$$= \left(\partial_\mu \partial^\mu\right) A^\nu - \partial^\nu \left(\partial_\mu A^\mu\right)$$

In the Lorenz gauge, this $= -\mu_0 J^\nu$

But Maxwell's Eqs. shouldn't depend on a particular choice for the gauge.

Let's look at $\partial_\mu F^{\mu\nu}$ term-by-term:

$$\partial_\mu F^{\mu 0} = \frac{\partial F^{00}}{\partial(ct)} + \frac{\partial F^{10}}{\partial x} + \frac{\partial F^{20}}{\partial y} + \frac{\partial F^{30}}{\partial z}$$

$$= 0 + \frac{\partial}{\partial x}\left(-E_x/c\right) + \frac{\partial}{\partial y}\left(-E_y/c\right) + \frac{\partial}{\partial z}\left(-E_z/c\right)$$

$$= -\frac{1}{c}\vec{\nabla}\cdot\vec{E} = -\frac{\rho}{c\varepsilon_0} = -\frac{\overbrace{(\rho c)}^{=J^0}}{c^2\varepsilon_0} = -\mu_0 J^0 \Rightarrow$$

Gauss' Law, independent of gauge!

$$\partial_\mu F^{\mu 1} = \frac{\partial F^{01}}{\partial(ct)} + \frac{\partial F^{11}}{\partial x} + \frac{\partial F^{21}}{\partial y} + \frac{\partial F^{31}}{\partial z}$$

$$= \frac{1}{c}\frac{\partial(E_x/c)}{\partial t} + 0 + \frac{\partial(-B_z)}{\partial y} + \frac{\partial(B_y)}{\partial z}$$

$$= \mu_0\varepsilon_0\frac{\partial E_x}{\partial t} - \left(\vec{\nabla}\times\vec{B}\right)_x$$

$$= -\mu_0 J_x = -\mu_0 J^1$$

Likewise for $\partial_\mu F^{\mu 2}$ and $\partial_\mu F^{\mu 3}$ $\rightarrow$

Maxwell-Ampere's Law, ind. of gauge!

$\Rightarrow$ $\boxed{\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu}$ gives the inhomogeneous Maxwell Eqs.

Griffiths writes $\partial_\nu F^{\mu\nu} = +\mu_0 J^\mu$: same thing.

For the homogeneous Maxwell Eqs., start by defining the Dual Tensor (usually written $\tilde{\mathcal{F}}$, but not in Griffiths)

$$\boxed{G^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}}$$

with $\varepsilon^{\mu\nu\rho\sigma}$ = 4-d Levi-Civita tensor

$$F_{\rho\sigma} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$G^{01} = \frac{1}{2}\left(\varepsilon^{0123}F_{23} + \varepsilon^{0132}F_{32}\right)$$

$$= \frac{1}{2}\left((+1)\,B_x + (-1)(-B_x)\right) = B_x$$

$$G^{02} = \frac{1}{2}\left(\varepsilon^{0213}F_{13} + \varepsilon^{0231}F_{31}\right)$$

$$= \frac{1}{2}\left((-1)(-B_y) + (+1)\,B_y\right) = B_y$$

Similarly, $G^{03} = B_z$, while $G^{i0} = -B_i$

$$G^{13} = \frac{1}{2}\left(\varepsilon^{1302}F_{02} + \varepsilon^{1320}F_{20}\right)$$

$$0123 \xrightarrow{(-1)} 0132 \xrightarrow{(-1)} 1032 \xrightarrow{(-1)} 1302 \Rightarrow$$

$$G^{13} = \frac{1}{2}\left((-1)\left(-\frac{E_y}{c}\right) + (+1)\left(\frac{E_y}{c}\right)\right) = \frac{E_y}{c}$$

$$G^{12} = \frac{1}{2}\left(\varepsilon^{1203} F_{03} + \varepsilon^{1230} F_{30}\right)$$

$$= \frac{1}{2}\left((+1)\left(-\frac{E_z}{c}\right) + (-1)\left(\frac{E_z}{c}\right)\right) = -\frac{E_z}{c}$$

$$G^{23} = \frac{1}{2}\left(\varepsilon^{2301} F_{01} + \varepsilon^{2310} F_{10}\right)$$

$0123 \rightarrow 0213 \rightarrow 2013 \rightarrow 2031 \rightarrow 2301 \quad (-1)^4$

$$G^{23} = \frac{1}{2}\left((+1)\left(-\frac{E_x}{c}\right) + (-1)\left(\frac{E_x}{c}\right)\right) = -\frac{E_x}{c}$$

Also $G^{31} = -\frac{E_y}{c}$, $G^{21} = \frac{E_z}{c}$, $G^{32} = \frac{E_x}{c} \Rightarrow$

$$\boxed{G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & -E_x/c \\ -B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{pmatrix}}$$

<u>Note</u>: From $F^{\mu\nu} \rightarrow G^{\mu\nu}$: $\frac{\vec{E}}{c} \rightarrow \vec{B}$; $\vec{B} \rightarrow -\frac{\vec{E}}{c}$

$$\partial_\mu G^{\mu 0} = 0 + \frac{\partial}{\partial x}(-B_x) + \frac{\partial}{\partial y}(-B_y) + \frac{\partial}{\partial z}(-B_z)$$

$$= -\vec{\nabla}\cdot\vec{B} = 0$$ Nobody's Law!

$$\partial_\mu G^{\mu 1} = \frac{\partial}{\partial(ct)}(B_x) + 0 + \frac{\partial}{\partial y}\left(E_z/c\right) + \frac{\partial}{\partial z}\left(-E_y/c\right)$$

$$= \frac{1}{c}\left[\frac{\partial B_x}{\partial t} + \left(\vec{\nabla}\times\vec{E}\right)_x\right] = 0$$

Similar for $\partial_\mu G^{\mu 2}$ and $\partial_\mu G^{\mu 3}$ $\Rightarrow$

$\partial_\mu G^{\mu i} = 0$ gives Faraday's Law!

$\Rightarrow$ $\boxed{\partial_\mu G^{\mu\nu} = 0}$ gives the homogeneous Maxwell Eqs.

Thus, $\boxed{\begin{array}{c} \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \\ \partial_\mu G^{\mu\nu} = 0 \end{array}}$ All four Maxwell's Eqs.

No potentials needed $\Rightarrow$ no constraints on gauge.

To have a complete description of electrodynamics, we also need the $\boxed{\text{Lorentz force law}}$:

$\vec{F} = \frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v}\times\vec{B}$ has ugly transformation rules.

The Minkowski force analog is:

$$\boxed{\frac{dp^\mu}{d\tau} = q\, F^{\mu\nu} \eta_\nu}$$

Proof: $\frac{dp^1}{d\tau} = \frac{dp^1}{dt}\overbrace{\frac{dt}{d\tau}}^{=\gamma_u} = q\left[F^{10}\eta_0 + F^{12}\eta_2 + F^{13}\eta_3\right]$

$$= q\left[\left(-\frac{E_x}{c}\right)(-\gamma_u c) + B_z(\gamma_u u_y) + (-B_y)(\gamma_u u_z)\right]$$

$$\Rightarrow \frac{dp_x}{dt} = qE_x + q(\vec{u}\times\vec{B})_x$$

Similar for $\frac{dp^2}{d\tau}$ and $\frac{dp^3}{dt}$ give Lorentz force law.

But what about the "extra" equation:

$$\frac{dp^0}{d\tau} = \frac{dp^0}{dt}\overbrace{\frac{dt}{d\tau}}^{=\gamma_u} = q\left[F^{01}\eta_1 + F^{02}\eta_2 + F^{03}\eta_3\right]$$

$$= q\left[\left(\frac{E_x}{c}\right)\gamma_u u_x + \left(\frac{E_y}{c}\right)\gamma_u u_y + \left(\frac{E_z}{c}\right)\gamma_u u_z\right]$$

But remember, $p^0 = \frac{\text{Energy}}{c} \Rightarrow$

$$\frac{d(\text{Energy})}{dt} = q\,\vec{u}\cdot\vec{E}$$ Standard result for power in EM field.