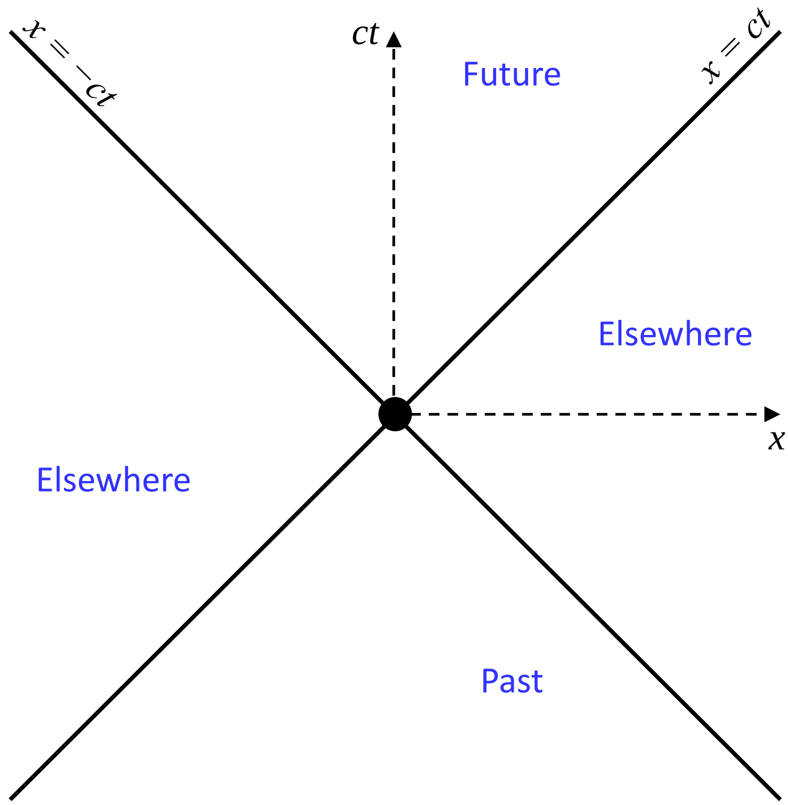
Space-Time Diagrams



- Future and the Past have "time-like" separations from the origin
  Space-time interval: I < 0 (with Griffiths sign convention)
- Elsewhere has "space-like" separation from the origin
  Space-time interval: I > 0 (with Griffiths sign convention)
- The diagonals have "light-like" separations from the origin
  Space-time interval: I = 0

Particle trajectories ("world lines") have slopes > 1 for $v < c$

Boosts move points along hyperbolic curves that have the diagonals as their asymptotes.

Some other important 4-vectors?

Obvious generalization of $\vec{v} = \frac{d\vec{r}}{dt}$ is $\frac{dx^\mu}{dt}$.

DOESN'T WORK!

Can't have both numerator and denominator transform and still transform like numerator alone.

$\Rightarrow$ If numerator is a 4-vector, denominator must be Lorentz invariant for ratio to be 4-vector

A trick! We have a Lorentz invariant that is "like" time – proper time $\Rightarrow$

Define 4-velocity $\boxed{\eta^\mu = \frac{dx^\mu}{d\tau}} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma\frac{dx^\mu}{dt}$

$$\Rightarrow \quad \eta^\mu = \begin{pmatrix} \gamma c \\ \gamma \vec{u} \end{pmatrix} = \begin{pmatrix} \frac{c}{\sqrt{1-\frac{u^2}{c^2}}} \\ \frac{\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \end{pmatrix}$$

where Griffiths uses $\vec{u}$ as his notation for the ordinary velocity of an object. (Not frame.) Sometimes $\gamma \to \gamma_u$ for clarity.

The square of a 4-vector is Lorentz invariant.

What about η^μ?

$$\boxed{\eta_\mu \eta^\mu} = -\gamma_u^2 c^2 + \gamma_u^2 u^2 = -\frac{c^2-u^2}{1-\frac{u^2}{c^2}} = -c^2 \frac{c^2-u^2}{c^2-u^2} \boxed{= -c^2}$$

Constant, independent of particle speed?!?

<u>YES</u>! Speed changes with boosts;

η^2 must be unchanged $\Rightarrow$ can't depend on the speed in any particular frame.

Generalization of $\vec{p} = m\vec{v}$:

$$\boxed{p^\mu = m\eta^\mu = \begin{pmatrix} \frac{mc}{\sqrt{1-\frac{u^2}{c^2}}} \\ \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \end{pmatrix}}$$

Space part of p^μ:

$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$ is the 3-momentum in Special Relativity.

It even obeys Newton's Law: $\vec{F} = \frac{d\vec{p}}{dt}$ ← ordinary t; not proper time

$\Rightarrow$ <u>Side note</u>: Transformation law for $\vec{F}$ is a mess!

What is $p^0 = \frac{mc}{\sqrt{1-u^2/c^2}}$?

Kinetic energy

Consider the non-relativistic limit:

$$p^0 \simeq \frac{mc}{1-\frac{1}{2}\frac{u^2}{c^2}} \simeq mc\left(1+\frac{1}{2}\frac{u^2}{c^2}\right) = mc + \frac{\boxed{\frac{1}{2}mu^2}}{c}$$

If we assign mc^2 = rest mass energy, $p^0 = \frac{mc^2 + KE}{c} = \frac{E}{c}$

$$\Rightarrow \boxed{p^\mu = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}}$$ Energy -and- Momentum } in one 4-vector!

$$p = m\eta \Rightarrow p^2 = m^2\eta^2 = -m^2c^2 = -\left(\frac{E}{c}\right)^2 + p^2$$

$$\Rightarrow m^2c^4 = E^2 - p^2c^2 \quad \text{or} \quad E^2 = p^2c^2 + m^2c^4$$

The separate laws of conservation of energy and momentum are combined as conservation of p^μ

Consider a scattering reaction: $e + p \rightarrow e + p$

Particles "1" "2" "3" "4"

Must have $p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$ in any given frame.

But can also define Lorentz invariants:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

Phase of an EM wave:

$\vec{k}\cdot\vec{x}-\omega t$ just counts crests. The counts (a number) is invariant. So:

$$-\omega t+\vec{k}\cdot\vec{x} = -(ct)\left(\frac{\omega}{c}\right)+\vec{x}\cdot\vec{k} = \text{invariant}$$

$$\Rightarrow \boxed{\begin{pmatrix}\omega/c\\ \vec{k}\end{pmatrix} \text{ is a 4 vector}}$$

Griffiths skips this one!

In QM, $\hbar\vec{k}=\vec{p}$ and $\hbar\omega=E \Rightarrow \begin{pmatrix}E/c\\ \vec{p}\end{pmatrix}=\begin{pmatrix}\hbar\omega/c\\ \hbar\vec{k}\end{pmatrix}$

Chapter 9

But $\left(\frac{\omega}{c}\right)^2-k^2=0$ for EM waves $\Rightarrow$

$\boxed{m=0 \text{ for photons}}$ and

$\boxed{E=pc}$ ← Relation from Chapter 9

Doppler Shift as a Lorentz boost:

$$\frac{\omega'}{c}=\gamma\left(\frac{\omega}{c}-\beta k_1\right)=\gamma\left(\frac{\omega}{c}-\vec{\beta}\cdot\vec{k}\right)$$

$$\Rightarrow \boxed{\omega'=\gamma\omega\left(1-\beta\cos\theta\right)}$$

$$\text{or } \boxed{E'=\gamma\left(1-\beta\cos\theta\right)E}$$

Change of photon direction:

$$\tan\theta' = \frac{\sin\theta'}{\cos\theta'} = \frac{\sqrt{k_y'^2 + k_z'^2}/\cancel{k'}}{k_x'/\cancel{k'}} = \frac{\sqrt{k_y^2+k_z^2}}{\gamma(k_x - \beta k^0)}$$

Note: $k^0 = \omega/c = k$

$$= \frac{\sqrt{k_y^2+k_z^2}/k^0}{\gamma\left(\frac{k_x}{k^0} - \beta\right)} = \boxed{\frac{\sin\theta}{\gamma(\cos\theta - \beta)}}$$

BTW: $\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$ transforms $S \to S'$.

For $S' \to S$, we "just" invert this. Not so easy!

But we get the transformation $S' \to S$ by interchanging primed and unprimed variables and

$v \to -v \Rightarrow \beta \to -\beta \Rightarrow$

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}$$

can be written down "by inspection"!

Charges and Currents

$\boxed{\text{Charge is a Lorentz invariant!}}$

Indirect argument:

Protons in He and U move at ~same speed.
Electrons in He and U have very different speeds.
But He and U atoms are both electrically neutral.

$\Rightarrow$ Q can't depend on motion.

What about charge density ρ and current density $\vec{J}$?

$$\rho = \frac{Q}{V} = \frac{Q}{dx\,dy\,dz} = \underbrace{\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}}_{\text{Length contraction!}} \underbrace{\frac{Q}{dx_0\,dy_0\,dz_0}}_{\text{in rest frame of } Q}$$

$\Rightarrow \rho$ transforms like time component of η^μ

$$\vec{J} = \rho\vec{u} = \frac{\vec{u}}{\sqrt{1-u^2/c^2}}\,\frac{Q}{dx_0\,dy_0\,dz_0}$$ transforms like space components of η^μ

Need to unify the units:

$$\boxed{J^\mu = \begin{pmatrix} \rho c \\ \rho\vec{u} \end{pmatrix} = \begin{pmatrix} \rho c \\ \vec{J} \end{pmatrix} \text{ is a 4-vector}}$$

The "4-current" or "4-current density"

Conservation of Charge:

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \Rightarrow \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\Rightarrow \frac{\partial(c\rho)}{\partial(ct)} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\Rightarrow \frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = 0$$

Note the sign!

$\left(\frac{\partial}{\partial x^0}\ \frac{\partial}{\partial x^1}\ \frac{\partial}{\partial x^2}\ \frac{\partial}{\partial x^3}\right)$ is NOT a contravariant 4-vector!

It does not transform like x^μ.

Rather, it transforms like $x_\mu \Rightarrow$

$$\boxed{\partial_\mu = \frac{\partial}{\partial x^\mu} \text{ is a covariant 4-vector}}$$

Continuity Equation reads $\boxed{\partial_\mu J^\mu = 0}$

Lorentz invariant!

$\partial^\mu = g^{\mu\nu}\partial_\nu = \frac{\partial}{\partial x_\mu}$ is contravariant 4-vector.

If ∂^μ is contravariant 4-vector and

∂_μ is covariant 4-vector $\Rightarrow$

$$\partial^\mu \partial_\mu = -\frac{\partial^2}{\partial(ct)^2} + \nabla^2 = \boxed{\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \text{ is Lorentz invariant}}$$

Often labeled $\Box^2$

the "d'Alembertian".

Maxwell's Equations in Potential Form

In Chapter 10, we found that:

$$\left(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\right) V = -\frac{\rho}{\varepsilon_0}$$

$$\left(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\right) \vec{A} = -\mu_0 \vec{J}$$

if $\quad \vec{\nabla} \cdot \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$

$$\Rightarrow \frac{1}{c^2}\frac{\partial V}{\partial t} + \vec{\nabla} \cdot \vec{A} = \frac{1}{c}\frac{\partial\left(\frac{V}{c}\right)}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$$

$$\Rightarrow \boxed{A^\mu = \begin{pmatrix} V/c \\ \vec{A} \end{pmatrix} \text{ is the 4-potential}}$$

gives the Lorenz ~~Lorentz~~ gauge condition : $\boxed{\partial_\mu A^\mu = 0}$

It's Lorentz invariant! Nice!

$$\left(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\right)\vec{A} = \Box^2 \vec{A} = -\mu_0 \vec{J}$$

$$\left(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\right)V = -\frac{\rho}{\varepsilon_0} \Rightarrow$$

$$\Box^2 \frac{V}{c} = -\frac{\rho}{\varepsilon_0 c} = -\frac{\rho c}{\varepsilon_0 c^2} = -\mu_0 J^0$$

$$\Rightarrow \boxed{\Box^2 A^\mu = -\mu_0 J^\mu}$$

$\Rightarrow$ The two $\boxed{}$ expressions above, when combined with definitions of $\vec{E}$ and $\vec{B}$, give all of Maxwell's Equations!