BUT....

β is relative to the earth

t is not! t is the twin time

The twin is accelerating $\Rightarrow$ t is not measured in an inertial frame. It's measured in a frame where the twin is always at rest. $\Rightarrow$

$$\boxed{t = \text{Twin Wrist Watch Frame !!!}}$$

Called "Proper Time", and denoted τ.

Proper time is our first Lorentz Invariant, meaning it is independent of our choice of frame.

Thus, the $\boxed{}$ equation above really should have been written:

$$\boxed{\frac{d\beta}{d\tau} = \frac{g}{c}\left(1-\beta^2\right)}.$$

$$\text{Then } \frac{d\beta}{dt} = \frac{d\beta}{d\tau}\,\frac{d\tau}{dt} = \left[\frac{g}{c}(1-\beta)^2\right]\left(\frac{1}{\gamma}\right) = \frac{g}{c}\left(1-\beta^2\right)^{3/2}$$

EARTH TIME

MOVING CLOCK FACTOR

$$\Rightarrow \frac{d\beta}{\left(1-\beta^2\right)^{3/2}} = \frac{g}{c}\,dt \Rightarrow \underbrace{\int_0^\beta \frac{d\beta}{\left(1-\beta^2\right)^{3/2}}}_{=\frac{\beta}{\sqrt{1-\beta^2}}} = \int_0^t \frac{g}{c}\,dt = \frac{g}{c}t$$

from my Int Tables

$$\Rightarrow \frac{\beta^2}{1-\beta^2} = \left(\frac{g}{c}t\right)^2$$

$$\Rightarrow \boxed{\beta = \frac{\frac{g}{c}t}{\sqrt{1+\left(\frac{g}{c}t\right)^2}}}$$

with "t" measured in the (inertial) earth frame.

$$x(t) = c\int_0^t \beta(t)\,dt = c\,\frac{c}{g}\int_0^t \frac{\frac{g}{c}t}{\sqrt{1+\left(\frac{g}{c}t\right)^2}}\,\frac{g}{c}\,dt$$

$$= \frac{c^2}{g}\int_0^{\frac{g}{c}t} \frac{y\,dy}{\sqrt{1+y^2}} = \frac{c^2}{g}\left[\sqrt{1+y^2}\right]_{y=0}^{\frac{g}{c}t}$$

$$\boxed{= \frac{c^2}{g}\left[\sqrt{1+\left(\frac{g}{c}t\right)^2} - 1\right]}$$

SHORT TIME LIMIT

$$\approx \frac{c^2}{g}\left[1 + \frac{1}{2}\frac{g^2t^2}{c^2} - 1\right] = \frac{1}{2}gt^2$$

A simplification:

$$\boxed{\text{"RAPIDITY"} \quad y \quad \text{with} \quad \gamma = \cosh(y)}$$

$\gamma^2 - \beta^2\gamma^2 = (1-\beta^2)\gamma^2 = \frac{1}{\gamma^2}\gamma^2 = 1$ and

$\cosh^2 y - \sinh^2 y = 1 \Rightarrow \quad \boxed{\beta\gamma = \sinh(y)}$

$\beta = \frac{\beta\gamma}{\gamma} = \frac{\sinh(y)}{\cosh(y)} \Rightarrow \quad \boxed{\beta = \tanh(y)}$

Velocity Addition, version 2:

$$\begin{pmatrix} x^{0''} \\ x^{1''} \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 \\ -\beta_2\gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh y_2 & -\sinh y_2 \\ -\sinh y_2 & \cosh y_2 \end{pmatrix} \begin{pmatrix} \cosh y_1 & -\sinh y_1 \\ -\sinh y_1 & \cosh y_1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh y_2 \cosh y_1 + \sinh y_2 \sinh y_1 & \cdots \\ -(\sinh y_2 \cosh y_1 + \cosh y_2 \sinh y_1 & \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(y_1+y_2) & -\sinh(y_1+y_2) \\ -\sinh(y_1+y_2) & \cosh(y_1+y_2) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \Rightarrow$$

$$\boxed{y_3 = y_1 + y_2}$$ LINEAR ADDITION!

"Comfortable Twin, try 2"

$$dy = \frac{g}{c} d\tau \Rightarrow y = \frac{g}{c}\tau \Rightarrow \beta = \tanh y = \tanh\left(\frac{g\tau}{c}\right)$$

Same as before. Lots easier!

$$\frac{dt}{d\tau} = \gamma = \cosh(y) = \cosh\left(\frac{g}{c}\tau\right) \Rightarrow$$

$$dt = \cosh\left(\frac{g}{c}\tau\right) d\tau \Rightarrow \boxed{t = \frac{c}{g}\sinh\left(\frac{g}{c}\tau\right)}$$

$$\Rightarrow \beta(t) = \tanh\left[\cancel{\frac{g}{c}}\cancel{\frac{c}{g}}\sinh^{-1}\left(\frac{gt}{c}\right)\right]$$

$$\text{But } \tanh(\alpha) = \frac{\sinh(\alpha)}{\cosh(\alpha)} = \frac{\sinh(\alpha)}{\sqrt{1+\sinh^2(\alpha)}} \Rightarrow$$

$$\boxed{\beta(t) = \frac{\frac{g}{c}t}{\sqrt{1+\left(\frac{g}{c}t\right)^2}}},$$ Same as before, no Int tables needed.

Now we can find $x(t)$ by integrating:

Note: Must do $\int dt$, **not** $\int d\tau$, because $\beta = \frac{1}{c}\frac{dx}{dt}$

$$x(t) = c\int_0^t \beta(t)\,dt = c\,\frac{c}{g}\int_0^t \frac{\frac{g}{c}t}{\sqrt{1+\left(\frac{gt}{c}\right)^2}}\,\frac{g}{c}\,dt$$

$$= \frac{c^2}{g}\int_0^{\frac{gt}{c}} \frac{y}{\sqrt{1+y^2}}\,dy = \frac{c^2}{g}\left[\sqrt{1+y^2}\right]_{y=0}^{\frac{gt}{c}} \Rightarrow$$

$$x(t) = \frac{c^2}{g}\left[\sqrt{1+\left(\frac{gt}{c}\right)^2} - 1\right]$$

Short time limit $= \frac{c^2}{g}\left[\left(1+\frac{1}{2}\frac{g^2t^2}{c^2}\right) - 1\right]$

$= \frac{1}{2}gt^2$ = Freshman Physics!

Back to Lorentz transformations:

$$\Lambda^{\mu}{}_{\nu} = 4\times4 \text{ matrix that changes frames,}$$

UPPER

LOWER

with μ, ν Greek indices that span 0, 1, 2, 3

(Roman letters $i, j, \ldots$ only span 1, 2, 3 and give vectors only in space, not spacetime)

$$\boxed{x^{\mu\prime} = \Lambda^{\mu}{}_{\nu} x^{\nu}}$$

Einstein summation notation

BUT only sum an upper index with a lower index, or you've made a mistake!

To shift an index up or down, use the:

METRIC TENSOR

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

This is $g^{\mu\nu}$ in Griffiths and in general relativity.

Nuclear and High Energy use:

$$\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$x_\mu = g_{\mu\nu} x^\nu$ gives

LOWER UPPER

NOTE!

$$x_0 = -ct$$
$$x_1 = x$$
$$x_2 = y$$
$$x_3 = z$$

WHY?

DEFINITIONS:

A CONTRAVARIANT 4-VECTOR A^μ:

Any A^μ that transforms like x^μ under Lorentz transformations

A COVARIANT 4-VECTOR A_μ:

Any A_μ that transforms like x_μ under Lorentz transformations.

Sidenote: 4-vectors are not written with arrows.

$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$ is the spatial 3-vector part of A^μ or A_μ.

$$\boxed{\begin{aligned} &\text{Dot product } A \cdot B = A^\mu B_\mu = A_\nu B^\nu \\ &= -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 \\ &\text{is Lorentz Invariant} \end{aligned}}$$

<u>Proof</u> (for a boost along x):

$$A^{\mu'}B'_{\mu} = -\left(\gamma A^0 - \beta\gamma A^1\right)\left(\gamma B^0 - \beta\gamma B^1\right) + \left(\gamma A^1 - \beta\gamma A^0\right)\left(\gamma B^1 - \beta\gamma B^0\right) + A^2B^2 + A^3B^3$$

$$= -\gamma^2\left[A^0B^0 - \cancel{\beta A^0B^1} - \cancel{\beta A^1B^0} + \beta^2 A^1B^1\right] + \gamma^2\left[A^1B^1 - \cancel{\beta A^1B^0} - \cancel{\beta A^0B^1} + \beta^2 A^0B^0\right] + A^2B^2 + A^3B^3$$

$$= -\gamma^2(1-\beta^2)A^0B^0 + \gamma^2(1-\beta^2)A^1B^1 + A^2B^2 + A^3B^3$$

$$= -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 \checkmark$$

x^2 — SQUARED! $= x \cdot x = x^\mu x_\mu$ Plays the role of the vector norm squared.

Linearity of Lorentz transformations means that, if $x_{(1)}$ and $x_{(2)}$ are two points in spacetime,

POINT LABELS
NOT LORENTZ INDICES

then $\Delta x = x_2 - x_1$ is also a 4-vector and

$$\boxed{I = \Delta x^\mu \Delta x_\mu = -c^2(\Delta t)^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \text{ is invariant}}$$

Griffiths notation for the "Space-time Interval"