Liénard-Wiechart Potentials

Point charge q moving along path $\vec{w}(t)$

$$\Rightarrow \rho = q\,\delta^{(3)}\left(\vec{r} - \vec{w}(t)\right)\;;\quad \vec{J} = q\,\vec{v}\,\delta^{(3)}\left(\vec{r} - \vec{w}(t)\right)$$

In principle, it's easy to plug into integrals for $V, \vec{A}$:

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,', t_r)}{\kappa}\, d\tau'$$

But t_r being a fn of $\vec{r}\,'$ complicates things!

Griffiths: geometric argument with moving trains to conclude the volume $d\tau'$ depends on $\vec{v}$

Better: Bite the bullet and look ahead to special relativity:

$$V \xrightarrow[\text{REL}]{\text{SPECIAL}} \frac{1}{4\pi\varepsilon_0}\int q\,\frac{\delta^{(3)}\left(\vec{r}\,' - \vec{w}(t')\right)\delta(t' - t_r)}{\kappa}\, dx'dy'dz'dt'$$

$\int dt'$ doesn't just set $t' = t_r$ because $t_r = t - \frac{\kappa}{c}$

depends on t' indirectly through $\kappa = \left|\vec{r} - \vec{w}(t')\right|$

Our time-dependent δ-fn is $\delta\left[t'-t+\frac{|\vec{r}-\vec{w}(t')|}{c}\right]$

$$= \delta\left(f(t')\right)$$

Example 1.15 (pg 48) showed $\delta(kx) = \frac{1}{|k|}\delta(x)$.

P332 generalized: If $f(x_0)=0$, then

$$\delta\left(f(x_0)\right) = \frac{1}{|f'(x_0)|}\delta(x-x_0).$$

For us: $|\vec{r}-\vec{w}(t')| = \sqrt{\left(\vec{r}-\vec{w}(t')\right)^2} \Rightarrow$

$$\frac{d}{dt'} = \frac{1}{2}\,\frac{2\left(\vec{r}-\vec{w}(t')\right)\cdot\left[-\vec{v}(t')\right]}{|\vec{r}-\vec{w}(t')|} = -\hat{\mathscr{r}}\cdot\vec{v}$$

$$\Rightarrow f'(t_r) = 1-\frac{\hat{\mathscr{r}}\cdot v}{c}$$

<u>Note</u>: Always $> 0 \Rightarrow$ can drop $|\ |$

Bottom line: $\int dt'$ sets $t'=t_r$ <u>AND</u> $*\ \frac{1}{1-\frac{\hat{\mathscr{r}}\cdot\vec{v}}{c}} \Rightarrow$

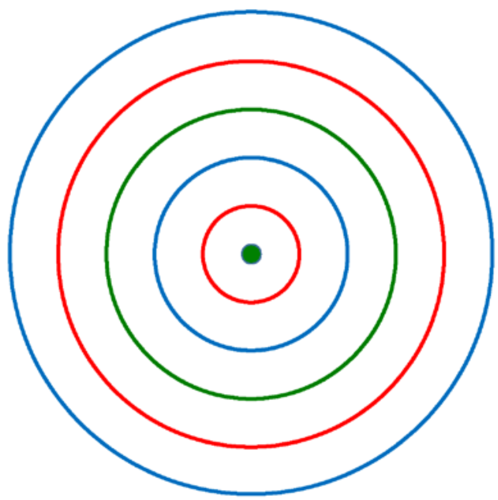
L-W POTENTIALS

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{\mathscr{r}\left[1-\frac{\hat{\mathscr{r}}\cdot\vec{v}}{c}\right]}} = \frac{1}{4\pi\varepsilon_0}\,\frac{qc}{\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v}} \qquad \text{Eq. 10.46}$$

$$\boxed{A = \frac{\mu_0}{4\pi}\,\frac{q\vec{v}}{\mathscr{r}\left[1-\frac{\hat{\mathscr{r}}\cdot\vec{v}}{c}\right]}} = \frac{\mu_0}{4\pi}\,\frac{qc\vec{v}}{\mathscr{r}c-\vec{\mathscr{r}}\cdot\vec{v}} = \frac{\vec{v}}{c^2}V \qquad \text{Eq. 10.47}$$

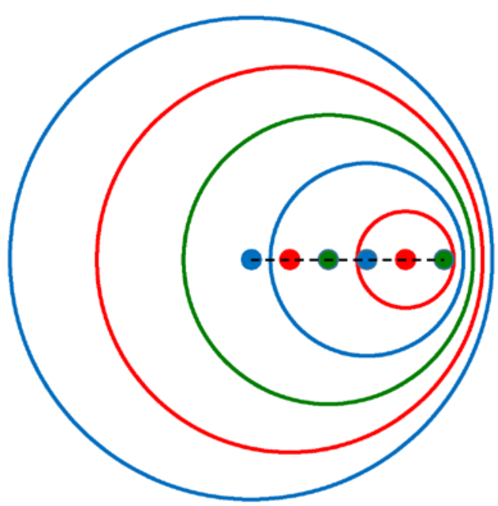
Signal front locations at time $5d/c$ emitted from a charged particle at rest at six equally spaced times:

$t = 0$ (large blue circle of radius $5d$)
$t = d/c$ (red circle of radius $4d$)
$t = 2d/c$ (green circle of radius $3d$)
$t = 3d/c$ (blue circle of radius $2d$)
$t = 4d/c$ (red circle of radius d)
$t = 5d/c$ (green point; seen at instant of emission)



Same signal fronts, but now emitted from a charged particle moving to the right at $v = 0.8c$:

The points indicate where the particle is located when a given signal is emitted.



WATCH OUT!

In the Liénard-Wiechart potentials, all quantities that depend on the particle motion:

$$\vec{r}' = \vec{w} \quad , \quad \vec{\mathscr{r}} \quad , \quad \vec{v}$$

are calculated at t_r, not at t.

Many texts use [ ] or $[\ ]_{ret}$ to make explicit which quantities are to be calculated at the retarded time. Griffiths leaves it for you to figure out on your own.

Furthermore, t_r is often not easy to calculate!

Example: Simplest possible case: $\vec{v} = \text{const} \Rightarrow$
$\vec{w} = \vec{v}\,t$
(at origin @ $t=0$)

$$|\vec{r} - \vec{w}(t_r)| = |\vec{r} - \vec{v}\,t_r| = c\,(t - t_r) \Rightarrow$$

Square both sides:

$$(\vec{r}-\vec{v}t_r)\cdot(\vec{r}-\vec{v}t_r) = c^2(t-t_r)^2$$

$$\Rightarrow r^2 - 2\vec{r}\cdot\vec{v}\,t_r + v^2 t_r^2 = c^2t^2 - 2c^2 t\,t_r + c^2 t_r^2$$

$$\Rightarrow (c^2-v^2)\,t_r^2 - 2(c^2t - \vec{r}\cdot\vec{v})\,t_r + (c^2t^2 - r^2) = 0$$

Quadratic formula:

$$t_r = \frac{(c^2t-\vec{r}\cdot\vec{v}) \pm \sqrt{(c^2t-\vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2-c^2t^2)}}{c^2-v^2}$$

Griffiths uses $\vec{v}=0$ to choose the sign.

I prefer:

(a) Squaring added a second solution

(b) That solution has "$t-t_r$" $<0 \Rightarrow$ "$t_r - t$" >0

(c) It's the advanced time solution!

We don't want it.

$\Rightarrow$ Must $\boxed{\text{adopt } - \text{ sign solution}}$ to our quadratic eq.

Consider the denominator in L-W potentials:

$$\mathscr{r}\left[1-\frac{\hat{\mathscr{r}}\cdot\vec{v}}{c}\right]=\overbrace{c(t-t_r)}^{\mathscr{r}}\left[1-\overbrace{\frac{\vec{r}-\vec{v}t_r}{c(t-t_r)}}^{\hat{\mathscr{r}}}\cdot\frac{\vec{v}}{c}\right]$$

$$=c(t-t_r)-\frac{\vec{r}\cdot\vec{v}}{c}+\frac{v^2}{c}t_r$$

$$=\frac{1}{c}\left[(c^2t-\vec{r}\cdot\vec{v})-(c^2-v^2)t_r\right]$$

FROM QUAD FORMULA

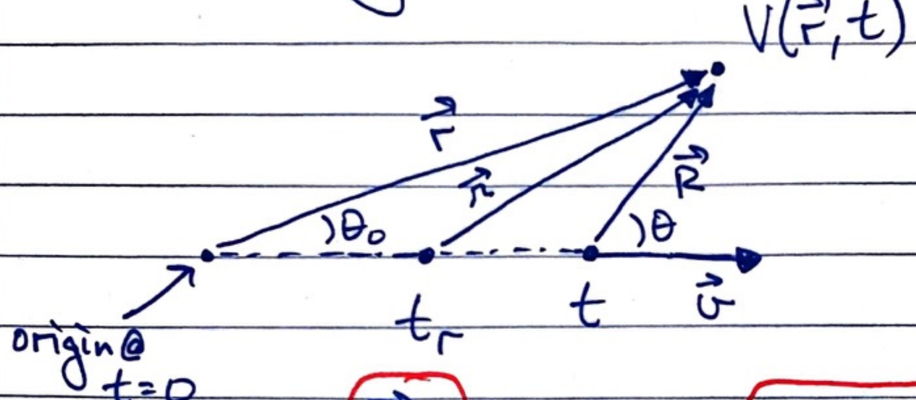
$$=\frac{1}{c}\sqrt{(c^2t-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2)}$$

$$\Rightarrow \boxed{\begin{aligned} V(\vec{r},t)&=\frac{1}{4\pi\varepsilon_0}\frac{qc}{\sqrt{(c^2t-\vec{r}\cdot\vec{v})^2+(c^2-v^2)(r^2-c^2t^2)}} && \text{Eq. 10.49}\\ \vec{A}(\vec{r},t)&=\mu_0\varepsilon_0\vec{v}V=\frac{\vec{v}}{c^2}V && \text{Eq. 10.50}\end{aligned}}$$

give V and $\vec{A}$ as functions of only of the coordinates and time where you want them.

Retarded time dependence has been removed, leaving expressions that look way more complicated than original L-W potentials.

A surprising simplification is possible in this constant velocity case:



Let $\boxed{\vec{R} =} \vec{r} - \vec{v}t$ be vector from current location

Then:

$$\left(c^2 t - \vec{r}\cdot\vec{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right)$$

$$= \cancel{c^4 t^2} - 2c^2 t\, \vec{r}\cdot\vec{v} + \left(\vec{r}\cdot\vec{v}\right)^2 + c^2 r^2 - v^2 r^2 - \cancel{c^4 t^2} + c^2 v^2 t^2$$

$$= c^2 \underbrace{\left[r^2 - 2\vec{r}\cdot\left(\vec{v}t\right) + v^2 t^2\right]}_{\left(\vec{r}-\vec{v}t\right)^2 = R^2} - \underbrace{\left[r^2 v^2 - \left(\vec{r}\cdot\vec{v}\right)^2\right]}_{r^2 v^2 \left(1 - \cos^2\theta_0\right)}$$

$$= r^2 v^2 \sin^2\theta_0$$

$$= R^2 v^2 \sin^2\theta$$

$$= R^2 c^2 \left(1 - \frac{v^2}{c^2}\sin^2\theta\right) \Rightarrow$$

$$\boxed{V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{R\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}} \quad \text{with } R + \theta \text{ from the current location}}$$