Section 10.4.4: Charged particle equations of motion in terms of potentials

Griffiths footnote says this can be skipped.

Instead we will expand the discussion!

Let $\vec{p}_{kin} = m\vec{v}$ be the kinetic momentum of a particle with charge q. Then:

$$\vec{F} = \frac{d\vec{p}_{kin}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

$$= q\left[-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} + \vec{v}\times\left(\vec{\nabla}\times\vec{A}\right)\right]$$

where we take $\vec{v} = \vec{v}(t)$, but independent of position,

consistent with the independent variables in Lagrangian mechanics. ($\Rightarrow \vec{\nabla}\times\vec{v} = 0;\ \vec{\nabla}\cdot\vec{v} = 0$)

From Prod Rule #4: $\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) - \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$

$$\Rightarrow \frac{d\vec{p}_{kin}}{dt} = q\left[-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} - \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A} + \vec{\nabla}\left(\vec{v}\cdot\vec{A}\right)\right]$$

But $\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial x}\frac{dx}{dt} + \frac{\partial\vec{A}}{\partial y}\frac{dy}{dt} + \frac{\partial\vec{A}}{\partial z}\frac{dz}{dt} + \frac{\partial\vec{A}}{\partial t}$

$$\boxed{= \left(\vec{v}\cdot\vec{\nabla}\right)A + \frac{\partial\vec{A}}{\partial t}}$$ Called convective derivative

$\Rightarrow$

$$\frac{d\vec{p}_{kin}}{dt} + q\frac{d\vec{A}}{dt} = \frac{d}{dt}\left(\vec{p}_{kin} + q\vec{A}\right) = -\vec{\nabla}\left(qV - q\vec{v}\cdot\vec{A}\right).$$

Griffiths defines $\boxed{U_{vel} = qV - q\vec{v}\cdot\vec{A}.}$

Implications / Applications?

Lagrangian $\boxed{L = T - U_{vel}} = \frac{1}{2}mv^2 - q\left(V - \vec{v}\cdot\vec{A}\right)$

$\Rightarrow \frac{\partial L}{\partial \dot{q}_i} = mv_i + qA_i = i^{th}$ component of canonical momentum

$\Rightarrow \boxed{\vec{p}_{can} = \vec{p}_{kin} + q\vec{A}}$ or $\boxed{\vec{p}_{kin} = \vec{p}_{can} - q\vec{A}}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \Rightarrow \frac{d\vec{p}_{can}}{dt} = \frac{d}{dt}\left(\vec{p}_{kin} + q\vec{A}\right) = -\vec{\nabla}U_{vel}. \checkmark$$

L gives the proper equation of motion.

Hamiltonian $H = \sum_i p_{i_{can}}\dot{q}_i - L$

$$= \sum_i \left(p_{kin_i} + qA_i\right)v_i - \frac{1}{2}mv^2 + qV - q\vec{v}\cdot\vec{A}$$

$$= mv^2 + q\vec{v}\cdot\vec{A} - \frac{1}{2}mv^2 + qV - q\vec{v}\cdot\vec{A}$$

$$= \frac{1}{2}mv^2 + qV = \boxed{\frac{\left(\vec{p}_{can} - q\vec{A}\right)^2}{2m} + qV}$$ with $\vec{r}$ + $\vec{p}_{can}$ as independent variables

Important in Quantum Mechanics where:

(a) The Schrodinger Eq. uses H

(b) $\vec{p}_{can} \to -i\hbar\vec{\nabla}$, not $\vec{p}_{kin}$

Hamilton's Equations:

$$\dot{q}_i = v_i = \frac{\partial H}{\partial p_{can_i}} \Rightarrow v_i = \frac{\cancel{2}(p_{can_i} - qA_i)}{\cancel{2}m}$$

$$\Rightarrow \boxed{\vec{v} = \frac{\vec{p}_{can} - q\vec{A}}{m}}, \text{ consistent with } \vec{p}_{kin} = \vec{p}_{can} - q\vec{A}$$

<u>Note</u>: $\vec{A} = \vec{A}(\vec{r}, t) \Rightarrow$ in Hamiltonian mechanics, the velocity depends on the canonical momentum <u>AND</u> (implicitly) on the position!

$$\dot{p}_{can_i} = -\frac{\partial H}{\partial q_i} \Rightarrow \frac{d\vec{p}_{can}}{dt} = -\vec{\nabla}H \Rightarrow$$

$$\frac{d\vec{p}_{can}}{dt} = -\vec{\nabla}(qV) - \frac{1}{2m}\vec{\nabla}\left[(\vec{p}_{can} - q\vec{A})\cdot(\vec{p}_{can} - q\vec{A})\right]$$

$$\overset{\text{PROD RULE \#4}}{=} -\vec{\nabla}(qV) - \frac{2}{2m}\Big[(\vec{p}_{can} - q\vec{A}) \times \{\vec{\nabla} \times (\vec{p}_{can} - q\vec{A})\} + \{(\vec{p}_{can} - q\vec{A})\cdot\vec{\nabla}\}(\vec{p}_{can} - q\vec{A})\Big]$$

But $\frac{2}{2m}(\vec{p}_{can} - q\vec{A}) = \vec{v}$, while $\vec{p}_{can}$ + $\vec{r}$ are independent in Hamiltonian mechanics, so $\vec{\nabla}$ on p_{can} gives zero.

Thus,

$$\frac{d\vec{p}_{can}}{dt} = -\vec{\nabla}(qV) - \left\{ \vec{v} \times (\vec{\nabla} \times (-q\vec{A})) + (\vec{v} \cdot \vec{\nabla})(-q\vec{A}) \right\}$$

$$= -\vec{\nabla}\left(qV - q\vec{v}\cdot\vec{A}\right)$$ from Prod Rule #4

where we have now interpreted $\vec{v}$ as independent of $\vec{\nabla}$, as appropriate for a Newtonian or Lagrangian Equation of Motion.

$$\Rightarrow \boxed{\frac{d\vec{p}_{can}}{dt} = -\vec{\nabla}U_{vel}}$$, as we found before.