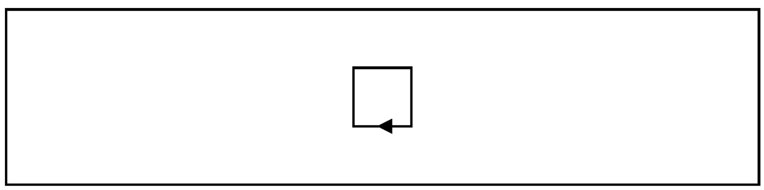
Physics 305 – Final Exam
Tuesday, May 5, 2020

There are four problems on this exam. Each problem is worth 25 points. When you are done, follow the instructions on the exam cover page to submit your papers to eCampus. Be sure to include a signed copy of the cover page (which was previously e-mailed to you, is posted on eCampus and the class web page, and is attached to this exam).

GOOD LUCK, HAVE A GREAT SUMMER, and STAY HEALTHY !!!

(1) (25 pts) A square loop of wire with sides of length a is centered within a rectangular loop of wire with size $L \times 3a$. Assume $L >> 3a$. A current $I(t) = kt$, where k is constant, is flowing around the square in the direction shown in the figure below. The rectangular loop has resistance R. Find the magnitude and direction of the current flowing in the rectangle.



(2) An infinite current sheet is located in the x-y plane. Assume (even though it's not physically possible!) that the surface current density throughout the entire sheet suddenly shifts from $\boldsymbol{K} = 0$ to $\boldsymbol{K} = K_0\,\hat{\boldsymbol{x}}$ at $t = 0$, and then remains constant thereafter. Also assume that this current is established in a manner that preserves the electrical neutrality of the sheet at all times.

a) (10 pts) What is the vector potential at an arbitrary point along the z axis at some later time t?

b) (7 pts) What are the electric and magnetic fields along the z axis at time t produced by the current sheet?

c) (8 pts) Calculate the Poynting vector for an arbitrary point along the z axis at time t. Give a physical explanation for your result in light of the fields you found in part (b).

(3) Consider a particle of mass m and charge q. At $t = 0$, the particle is at rest at the origin. At that moment, a constant electric field $\boldsymbol{E} = E_x\,\hat{\boldsymbol{x}}$ turns on.

a) (17 pts) Find parametric equations that describe the subsequent motion of the particle.

b) (8 pts) Solve your parametric equations to find the position of the particle as a function of time.
Note: It's possible to do part (b) without doing part (a) first, though I personally think that approach is more difficult. If you find the particle position as a function of time without going the parametric route, you will receive full credit for this problem.

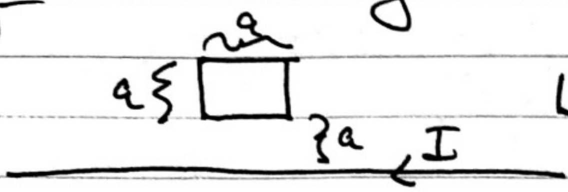
(4) Consider a particle of mass M, initially at rest, which decays into a final state that contains two particles with masses m_a and m_b.

a) (15 pts) Calculate the energy of particle a in the final state.

b) (10 pts) Now assume that M decays into a final state that contains four particles, with masses m_1, m_2, m_3, and m_4. Calculate the *maximum* energy of particle 1 in the final state.
Hint: Don't do a new calculation for part (b). Use your result from part (a).

Final Exam, Problem 1

Calculate the mutual inductance by putting I in the outer rectangle and calculating Φ through the square:



$L \gg 3a \Rightarrow$ can use B for a line current

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi} \Rightarrow \Phi = \frac{\mu_0 I a}{2\pi}\int_a^{2a}\frac{ds}{s} = \frac{\mu_0 I a}{2\pi}\ln(2) = MI.$$

Second side of rectangle doubles this $\Rightarrow \boxed{M = \frac{\mu_0 a}{\pi}\ln(2)}$.

Then $\mathcal{E} = M\frac{dI}{dt} = \frac{\mu_0 a k}{\pi}\ln(2) \Rightarrow$

$$\boxed{I = \frac{\mathcal{E}}{R} = \frac{\mu_0 a k}{\pi R}\ln(2)}$$

The induced B field in the large rectangle will try to oppose the flux change in the small square. Thus, it will flow $\boxed{\text{counterclockwise}}$.

Final Exam, Problem 2

(a) $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\,d\tau' = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,',t_r)}{\mathscr{r}}\,da'$. At a point z,

$t_r = t - \frac{\sqrt{z^2+s^2}}{c}$ must be $\geq 0 \Rightarrow ct \geq \sqrt{z^2+s^2} \Rightarrow s \leq \sqrt{(ct)^2-z^2}$.

This gives $\vec{A} = \frac{\mu_0 K_0 \hat{x}}{4\pi}(2\pi)\int_0^{\sqrt{(ct)^2-z^2}} \frac{s\,ds}{\sqrt{s^2+z^2}}$ (Note: 2π is $\int_0^{2\pi} d\varphi$)

$= \frac{\mu_0 K_0}{2}\hat{x}\left[\sqrt{s^2+z^2}\right]_{s=0}^{\sqrt{(ct)^2-z^2}} = \frac{\mu_0 K_0}{2}\hat{x}\left[ct-|z|\right]$

$\Rightarrow \boxed{\vec{A} = \frac{\mu_0 K_0}{2}\hat{x}\,(ct-|z|)\,\theta(ct-|z|)}$.

(b) $V=0 \Rightarrow \vec{E} = -\frac{\partial\vec{A}}{\partial t} = \boxed{-\frac{\mu_0 K_0 c}{2}\,\theta(ct-|z|)\,\hat{x}}$

$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\partial A_x}{\partial z}\hat{y} = \boxed{-\frac{\mu_0 K_0}{2}\frac{z}{|z|}\,\theta(ct-|z|)\,\hat{y}}$

(c) $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\left[\left(-\frac{\mu_0 K_0 c}{2}\theta(ct-|z|)\hat{x}\right)\times\left(-\frac{\mu_0 K_0}{2}\frac{z}{|z|}\theta(ct-|z|)\hat{y}\right)\right]$

$= \frac{\mu_0 K_0^2 c}{4}\frac{z}{|z|}\,\theta(ct-|z|)\,\hat{z}$

$\vec{S}$ always points away from the x-y plane. It represents energy flowing away from the plane to establish the $\vec{E}$ + $\vec{B}$ fields at ever larger distances $|z|$ from the x-y plane.

Final Exam, Problem 3

(a) $\frac{dp^\mu}{d\tau} = m\frac{du^\mu}{d\tau} = qF^{\mu\nu}u_\nu$ with $F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & 0 & 0 \\ -\frac{E_x}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

the motion is purely in x, so:

$\frac{du^0}{d\tau} = \frac{q}{m}F^{01}u_1 = \frac{q}{m}\frac{E_x}{c}u_1 = \frac{q}{m}\frac{E_x}{c}u^1$ and

$\frac{du^1}{d\tau} = \frac{q}{m}F^{10}u_0 = \frac{q}{m}\left(-\frac{E_x}{c}\right)u_0 = \frac{q}{m}\frac{E_x}{c}u^0$. $\boxed{\text{Let } \alpha = \frac{q}{m}\frac{E_x}{c}}$

Combine to give: $\frac{d^2u^0}{d\tau^2} - \alpha^2 u^0 = 0$, $\frac{d^2u^1}{d\tau^2} - \alpha^2 u^1 = 0$.

Take $\tau = 0$ at $t = 0 \Rightarrow u^0(\tau=0) = \gamma c = c$, $u^1(\tau=0) = \gamma v = 0$

$\frac{du^0}{d\tau}(\tau=0) = 0$, $\frac{du^1}{d\tau}(\tau=0) = \alpha c$.

$\Rightarrow \boxed{u^0 = c\cosh(\alpha\tau), \quad u^1 = c\sinh(\alpha\tau)}$

$u^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = c\cosh(\alpha\tau) \Rightarrow t = \frac{\sinh(\alpha\tau)}{\alpha}$

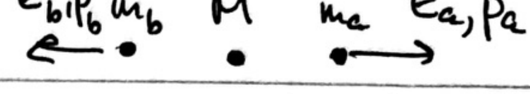
$u^1 = \frac{dx^1}{d\tau} = c\sinh(\alpha\tau) \Rightarrow x = \frac{c}{\alpha}\left[\cosh(\alpha\tau) - 1\right]$

(b) $\alpha t = \sinh(\alpha\tau) \Rightarrow 1 + \sinh^2(\alpha\tau) = \cosh^2(\alpha\tau) = 1 + (\alpha t)^2$

$\Rightarrow$

$$\boxed{\begin{array}{l} x(t) = \frac{c}{\alpha}\left[\sqrt{1+(\alpha t)^2} - 1\right] \text{ with } \alpha = \frac{q}{m}\frac{E_x}{c} \\ y(t) = z(t) = 0 \text{ for all } t \end{array}}$$

Final Exam, Problem 4



(a) Conservation of energy gives $\boxed{Mc^2 = E_a + E_b}$ ①.

Conservation of momentum gives $p_a = p_b \Rightarrow \sqrt{\left(\frac{E_a}{c}\right)^2 - m_a^2c^2} = \sqrt{\left(\frac{E_b}{c}\right)^2 - m_b^2c^2}$

$\Rightarrow \frac{E_a^2}{c^2} - m_a^2c^2 = \frac{E_b^2}{c^2} - m_b^2c^2 \Rightarrow \boxed{E_a^2 - E_b^2 = (m_a^2 - m_b^2)c^4}$ ②

$\frac{\text{Eq. ②}}{\text{Eq. ①}} \Rightarrow \boxed{E_a - E_b = \frac{m_a^2 - m_b^2}{M}c^2}$ ③

Then ① + ③ $\Rightarrow 2E_a = \frac{m_a^2 - m_b^2}{M}c^2 + Mc^2$

$$\Rightarrow \boxed{E_a = \frac{M^2 + m_a^2 - m_b^2}{2M}c^2}.$$

(b) For the 4-particle decay, we have $p_M^\mu = p_{m_1}^\mu + p_{m_2}^\mu + p_{m_3}^\mu + p_{m_4}^\mu$
$= p_{m_1}^\mu + \left(p_{m_2}^\mu + p_{m_3}^\mu + p_{m_4}^\mu\right)$. So if we think of the process as a pseudo-two-body decay $M \to m_1 + (m_2 + m_3 + m_4)$, then part (a) gives:

$$E_1 = \frac{M^2 + m_1^2 - m_{234}^2}{2M}c^2.$$

The maximum E_1 will occur when m_{234} = minimum
$= m_2 + m_3 + m_4 \Rightarrow$

$$\boxed{E_{1_{MAX}} = \frac{M^2 + m_1^2 - (m_2 + m_3 + m_4)^2}{2M}c^2}$$