Physics 305 – Exam 2

Thursday, April 16, 2020

There are four problems on this exam. Each problem is worth 25 points. Start each problem on a new sheet of paper, and use only one side of each sheet. When you are done, follow the instructions on the exam cover page to submit your papers to eCampus. Be sure to include a signed copy of the cover page (which was previously e-mailed to, is posted on eCampus and the class web page, and is attached to this exam).

GOOD LUCK !!!

(1) (25 pts) A thin slab of material has parallel surfaces. Light incident from air strikes the slab at Brewster's angle. Show that the refracted wave also strikes the second interface (*i.e.*, from inside the slab back to air) at Brewster's angle. Note: You may assume that $\mu_1 = \mu_2 = \mu_0$.

(2) A plane electromagnetic wave with $\vec{E} = E_0 \hat{x} \cos(kz - \omega t)$ interacts with an electron that is bound to the origin of the coordinate system by a harmonic restoring force $\vec{F} = -m\omega_0^{\,2} \vec{r}$. You may ignore any damping force felt by the electron. Also, assume that E_0 is "not too large". Calculate the time-averaged angular distribution of the scattered radiation and the total radiated power for two limiting cases (17 pts for the first case, plus 3 for the second):

a) $\omega_0 << \omega$. (This limit would be appropriate for a nearly free electron in a plasma.)

b) $\omega_0 >> \omega$. (This limit would be appropriate for a tightly bound atomic electron.)

c) (5 pts) Explain what would be necessary for E_0 to be "too large" for your answers in parts (a) and (b) to be valid?

(3) A time-dependent point charge $q(t)$ at the origin, $\rho(\vec{r},t) = q(t)\delta^{(3)}(\vec{r})$, is fed by a current $\vec{J}(\vec{r},t) = \dfrac{-1}{4\pi r^2}\dfrac{dq}{dt}\hat{r}$.

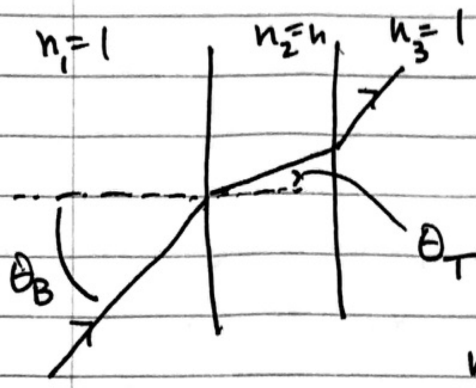
a) (10 pts) Find the scalar and vector potentials in the Coulomb gauge.

b) (5 pts) Find the electric and magnetic fields.

c) (10 pts) Show that your answers in part (b) satisfy Maxwell's equations.

(4) (25 pts) A hollow rectangular wave guide, which propagates waves in the $+z$ direction, has perfect conducting walls at $x = 0$, $x = a$, $y = 0$, and $y = b$, where $a > b$. A TM_{11} wave, with angular frequency ω, is propagating down the wave guide. The maximum value of the z component of the electric field within the wave is E_0 (which you can assume to be real valued). Calculate the surface current flowing along the wall at $y = 0$ as a function of position and time.

Exam 2, Problem 1



Brewster's angle obeys $\tan\theta_B = \frac{n_2}{n_1} = n$ at the front surface $\Rightarrow$

$$n = \frac{\sin\theta_B}{\sqrt{1-\sin^2\theta_B}} \Rightarrow n^2\left(1-\sin^2\theta_B\right) = \sin^2\theta_B$$

$\Rightarrow \sin^2\theta_B = \frac{n^2}{1+n^2} \Rightarrow \boxed{\sin\theta_B = \frac{n}{\sqrt{1+n^2}}}$. Meanwhile, Snell's Law gives $(1)\sin\theta_B = n\sin\theta_T \Rightarrow \boxed{\sin\theta_T = \frac{1}{\sqrt{1+n^2}}}$.

$$\Rightarrow \cos^2\theta_T = 1-\sin^2\theta_T = 1-\frac{1}{1+n^2} = \frac{n^2}{1+n^2} \Rightarrow$$

$$\boxed{\cos\theta_T = \frac{n}{\sqrt{1+n^2}}} \Rightarrow \tan\theta_T = \frac{\sin\theta_T}{\cos\theta_T} = \frac{1/\sqrt{1+n^2}}{n/\sqrt{1+n^2}} = \frac{1}{n} = \frac{n_3}{n_2}$$

$\Rightarrow$ θ_T is Brewster's angle at the second surface.

Alternative proof: $n_1\sin\theta_1 = n_2\sin\theta_2 = n_1\sin\theta_3 \Rightarrow$ $\theta_1 = \theta_3$. Meanwhile $\theta_1 = \theta_B \Rightarrow \tan\theta_1 = \frac{n_2}{n_1} = \frac{\sin\theta_1}{\cos\theta_1} \Rightarrow$

$n_1\sin\theta_1 = n_2\cos\theta_1 \Rightarrow \cos\theta_1 = \sin\theta_2 \Rightarrow \cos\theta_2 = \sin\theta_1$

and $\tan\theta_2 = \frac{\sin\theta_2}{\cos\theta_2} = \frac{\cos\theta_1}{\sin\theta_1} = \frac{n_1}{n_2}$. Thus, θ_2 is at Brewster's angle at the second surface.

Exam 2, Problem 2

$\vec{F} = m\frac{d^2\vec{r}}{dt^2} = -m\omega_0^2\vec{r} - eE_0\hat{x}\cos(\omega t) \Rightarrow$ oscillations in x, so

let $x(t) = \mathrm{Re}\left[x_0 e^{-i\omega t}\right] \Rightarrow -m\omega^2 x_0 = -m\omega_0^2 x_0 - eE_0 \Rightarrow$

$x_0 = \frac{-eE_0}{m(\omega_0^2-\omega^2)} \Rightarrow \vec{p}(t) = \frac{e^2E_0}{m(\omega_0^2-\omega^2)}\hat{x}\cos(\omega t)$. The time-

averaged ang dist is $\frac{dP}{d\Omega} = \langle\vec{S}\rangle\cdot r^2\hat{r} = \frac{\mu_0}{16\pi^2 c}\langle\ddot{p}(t_0)^2\rangle \sin^2\theta_{\hat{r}-\hat{x}}$

$\langle\ddot{p}(t_0)^2\rangle = \left\langle\left[\frac{-e^2E_0\omega^2}{m(\omega_0^2-\omega^2)}\cos(\omega t)\right]^2\right\rangle = \frac{e^4E_0^2\omega^4}{2m^2(\omega_0^2-\omega^2)^2}$, and

$\sin^2\theta_{\hat{r}\cdot\hat{x}} = 1-\cos^2\theta_{\hat{r}-\hat{x}} = 1-\sin^2\theta\cos^2\varphi \Rightarrow$

$\boxed{\frac{dP}{d\Omega} = \frac{\mu_0 e^4 E_0^2\omega^4}{32\pi^2 c m^2(\omega_0^2-\omega^2)^2}\left(1-\sin^2\theta\cos^2\varphi\right)}$, and the total

power radiated $= P = \frac{\mu_0}{6\pi c}\langle\ddot{p}(t_0)^2\rangle = \boxed{\frac{\mu_0 e^4 E_0^2\omega^4}{12\pi c m^2(\omega_0^2-\omega^2)^2}}$

(a) $\omega_0 \ll \omega \Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{\mu_0 e^4 E_0^2}{32\pi^2 c m^2}\left(1-\sin^2\theta\cos^2\varphi\right);\ P_{tot} = \frac{\mu_0 e^4 E_0^2}{12\pi c m^2}}$

(b) $\omega_0 \gg \omega \Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{\mu_0 e^4 E_0^2}{32\pi^2 c m^2}\left(\frac{\omega}{\omega_0}\right)^4\left(1-\sin^2\theta\cos^2\varphi\right);\ P_{tot} = \frac{\mu_0 e^4 E_0^2}{12\pi c m^2}\left(\frac{\omega}{\omega_0}\right)^4}$

(c) Above all require the long-wavelength approximation $\Rightarrow$

we need $|x_0| \ll \frac{c}{\omega} \Rightarrow \frac{eE_0}{m|\omega_0^2-\omega^2|} \ll \frac{c}{\omega} \Rightarrow$

we will get in trouble if E_0 approaches $\boxed{E_0 \sim \frac{mc|\omega_0^2-\omega^2|}{e\omega}}$.

Exam 2, Problem 3

a) In the Coulomb gauge, $V(\vec{r},t) = \frac{1}{4\pi\varepsilon}\int \frac{\rho(\vec{r}',t)}{\mathscr{r}} d\tau' \Rightarrow$

$\boxed{V = \frac{1}{4\pi\varepsilon_0}\frac{q(t)}{r}}$. Spherical symmetry $\Rightarrow$ we expect $\vec{B}=0$ (will prove in part c) $\Rightarrow \vec{\nabla}\times\vec{A}=0$. But $\vec{\nabla}\cdot\vec{A}=0$ for Coulomb gauge, and $\vec{A}\to 0$ as $\vec{r}\to\infty \Rightarrow$ $\boxed{\vec{A}=0 \text{ everywhere}}$.

(b) Already said $\boxed{\vec{B}=0}$. $\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V = \boxed{\frac{1}{4\pi\varepsilon_0}\frac{q(t)}{r^2}\hat{r}}$

(c) At any time t, ρ and $\vec{E}$ are given by the static ρ and $\vec{E}$ for a point charge at the origin $\Rightarrow$

$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{E}_{static} = \frac{\rho_{static}}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} \Rightarrow$ Gauss' Law okay.

$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\vec{E}_{static} = 0 = -\frac{\partial\vec{B}}{\partial t} \Rightarrow$ Faraday's Law okay

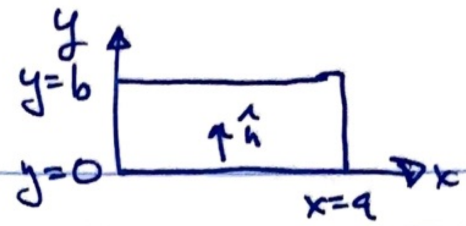
Also, $\vec{B}=0$ everywhere $\Rightarrow \vec{\nabla}\cdot\vec{B}=0 \Rightarrow$ No One's Law okay

Finally, $\vec{\nabla}\times\vec{B}=0$. But $\mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

$= \mu_0\left(\frac{-1}{4\pi r^2}\frac{dq}{dt}\hat{r}\right) + \mu_0\varepsilon_0\left[\frac{1}{4\pi\varepsilon_0}\frac{dq/dt}{r^2}\hat{r}\right] = 0 \Rightarrow$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \Rightarrow$ Maxwell-Ampère's Law okay

Exam 2, Problem 4



$\vec{H}=0$ in the conductor so the surface current obeys $\vec{H}_{\parallel} = \vec{K}_f \times \hat{n}$, where $H_{\parallel}$ is the tangential field at the surface and $\hat{n} = \hat{y}$ points into the wave guide at $y=0$.

$B_z = 0$ for a TM_{11} wave, so we need $B_x(x)$ on $y=0$.

In a TM_{11} wave, $E_z = E_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) e^{i(kz-\omega t)}$ with $\left(\frac{\omega}{c}\right)^2 - k^2 = \frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}$. I will drop $e^{i(kz-\omega t)}$ till the end.

$$B_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(\frac{-\omega}{c^2}\frac{\partial E_z}{\partial y}\right) = \frac{-i \quad \omega/c^2}{\pi^{\cancel{2}}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)} E_0 \sin\left(\frac{\pi x}{a}\right)\frac{\cancel{\pi}}{b}\cos\left(\frac{\pi y}{b}\right) \Rightarrow$$

At $y=0$, $B_x = \frac{-i}{\pi b\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}\frac{E_0\omega}{c^2}\sin\left(\frac{\pi x}{a}\right) \Rightarrow$

$$\vec{H}_{\parallel} = \frac{1}{\mu_0}B_x\hat{x} = \frac{1}{\pi b\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}\frac{E_0\omega}{\mu_0 c^2}\sin\left(\frac{\pi x}{a}\right) \mathrm{Re}\left[-ie^{i(kz-\omega t)}\right]\hat{x}.$$

$\mathrm{Re}\left[-ie^{i(kz-\omega t)}\right] = \sin(kz-\omega t)$, and

$\vec{H}_{\parallel}$ in $\hat{x}$ and $\hat{n}$ in $\hat{y} \Rightarrow \vec{K}_f$ points in the $-\hat{z}$ direction

$$\Rightarrow \boxed{\vec{K}_f = \frac{-1}{\pi b\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}\frac{E_0\omega}{\mu_0 c^2}\sin\left(\frac{\pi x}{a}\right)\sin(kz-\omega t)\,\hat{z}}$$