Physics 305 – Exam 1

Thursday, February 20, 2020

There are four problems on this exam. Each problem is worth 25 points. Start each problem on a new sheet of paper, use only one side of each sheet, and encircle your answer(s) to each part. **GOOD LUCK !!!**

(1) An axial, cylindrically symmetric magnetic field is given by $\vec{B} = B(s)\hat{z}$. Initially, an ion of charge q and mass m moves under the influence of this magnetic field in a circular orbit in the x-y plane at a distance R from the z axis. The magnetic field is slowly increased in magnitude, while keeping the shape of the field (as a function of s) unchanged. Assume that the function $B(s)$ has the same sign everywhere and that the ion's motion is non-relativistic at all times.

a. (10 pts) Show that the ion speeds up.

b. (15 pts) Now assume that the ion remains in the same circular orbit, with radius R, as it speeds up. Find the average value of the magnetic field inside the ion's orbit, as a function of $B(R)$.
Note: This was the principle behind an early electron accelerator called the *Betatron*.

(2) Consider an infinitely long wire located on the z axis. There is a current in the wire, $I(z)$, which is a function of z, but not of t. In addition, there is a charge density along the wire, $\lambda(t)$, which is a function of t, but not of z. Assume that $I(z = 0) = 0$, and $\lambda(t = 0) = 0$.

a. (9 pts) Show that $\lambda(t) = kt$ and $I(z) = -kz$, where k is some constant.

b. (6 pts) Write down expressions for the electric and magnetic fields due to the charge and current of the wire in the quasi-static approximation.

c. (10 pts) Show that your expressions from part (*b*) obey Maxwell's equations, and thus represent the exact solution for all points in space at all times.
Warning: Be sure to address appropriately any δ-function cases that arise.

(3) An infinitely long, straight wire carries a current I along the z axis (in the $+z$ direction). It is surrounded by a thin, infinitely long, cylindrical conductor of radius R that carries an equal current I in the $-z$ direction. The region in between the wire and the cylinder is filled with isotropic, paramagnetic, non-conducting material with relative permeability μ_r.

a. (12 pts) Find **B**, **H**, and **M** at all points in space.

b. (10 pts) Find the vector potential at all points in space.

c. (3 pts) Verify that your result obeys $\nabla\cdot\mathbf{A} = 0$.

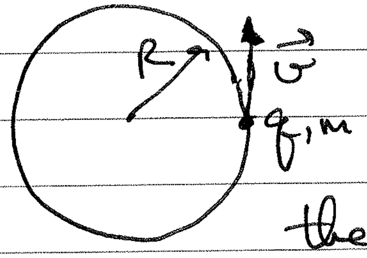
(4) A point charge q is located at rest at the center of a toroid that has an average radius a and a rectangular cross section, with height h and width w. The toroid consists of N tightly-wound turns carrying a current I. Assume $h \ll a$ and $w \ll a$.

a. (12 pts) Find the electromagnetic momentum and angular momentum densities at all points in space.

b. (4 pts) Find the total electromagnetic momentum and angular momentum for this configuration.

c. (9 pts) Now the current in the toroid is turned off, quickly enough so that the point charge does not move appreciably during the time the magnetic field drops to zero. Calculate the impulse imparted to the point charge q.

Exam 1, Problem 1

(a) 

$\vec{F} = q\vec{v}\times\vec{B}$ must point toward the origin for circular motion. Thus, if q is positive (negative), $\vec{B}$ must point into (out of) the paper. This means the magnetic flux from q through the circle points opposite to $\vec{B}$. When $\vec{B}$ increases in magnitude, Lenz's Law says an EMF ~~says an~~ will be induced that will try to inhibit the change in flux by producing an opposing flux. Increased opposing flux $\Rightarrow$ q speeds up.

(b) (a) gave* us the signs. Now let's just worry about magnitudes.

For circular motion, $\frac{mv^2}{R} = qvB \Rightarrow mv = p = qRB$. To stay in circular motion, this must stay true $\Rightarrow \frac{dp}{dt} = qR\frac{dB(R)}{dt}$.

But $\frac{dp}{dt} = qE$ where $\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = 2\pi R E(R) = (-)\frac{d\Phi}{dt}$

$\Rightarrow 2\cancel{\pi}\cancel{R}E(R) = \cancel{\pi}R^2\frac{d\langle B\rangle}{dt} \Rightarrow E(R) = \frac{R}{2}\frac{d\langle B\rangle}{dt}$. Thus,

$\frac{\cancel{q}\cancel{R}}{2}\frac{d\langle B\rangle}{dt} = \cancel{q}\cancel{R}\frac{dB(R)}{dt}$ at all times. $\Rightarrow$

$$\boxed{\langle B\rangle_{s<R} = 2B(R)}.$$

Exam 1, Problem 2

(a) The continuity equation requires $\frac{d\lambda}{dt} = -\frac{dI}{dz}$. This is a separated diff. eq., so both sides must equal the same constant $k \Rightarrow \lambda = kt + \lambda_0$, $I = -kz + I_0$. But $\lambda(t=0) = 0 \Rightarrow \lambda_0 = 0$ and $I(z=0) = 0 \Rightarrow I_0 = 0 \Rightarrow$

$\lambda = kt$ $\quad$ $I = -kz$ as desired.

(b) For a line charge, $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s} \Rightarrow \boxed{\vec{E} = \frac{kt}{2\pi\varepsilon_0 s}\hat{s}}$

For a line current, $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \Rightarrow \boxed{B = \frac{-\mu_0 kz}{2\pi s}\hat{\phi}}$

(c) At any instant in time, $\vec{E}$ obeys Gauss' Law from electrostatics with zero curl $\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$ and $\boxed{\vec{\nabla}\times\vec{E} = 0 = -\frac{\partial B}{\partial t}}$.

$\vec{B}$ points in the $\hat{\phi}$ direction, but has no ϕ-dep $\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$.

For $s \neq 0$, $\vec{\nabla}\times\vec{B} = -\frac{\partial B_\phi}{\partial z}\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}(sB_\phi)\hat{z} = \frac{\mu_0 k}{2\pi s}\hat{s} + 0$

$\mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \overset{s\neq 0}{=} \mu_0\cancel{\varepsilon_0}\frac{k}{2\pi\cancel{\varepsilon_0}s}\hat{s} = \frac{\mu k}{2\pi s}\hat{s} \Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}$.

For $s=0$, consider a circular loop around the z axis:

$\oint\vec{B}\cdot d\vec{l} = \frac{-\mu_0 kz}{\cancel{2\pi s}}\cancel{2\pi s} = \mu_0(-kz) = \mu_0 I(z)$.

Meanwhile, $\mu_0\varepsilon_0\int\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a} = 0$ because $\frac{\partial\vec{E}}{\partial t} \perp \vec{a}$. Thus, the integral form of Maxwell-Ampere's Law is satisfied at $s=0$ because the $\boxed{\delta\text{-fn in } \vec{\nabla}\times\vec{B} \text{ just matches } \mu_0 I(z)\hat{z}}$.

Exam 1, Problem 3

(a) All current points in the $\pm\hat{z}$ direction and is cylindrically symmetric and independent of z. Thus, $\vec{H} = H_\varphi \hat{\varphi}$ and we can use the integral form of Ampère's Law:

Inside: $\oint \vec{H}\cdot d\vec{l} = H_\varphi 2\pi s = I \Rightarrow \boxed{\vec{H} = \frac{I}{2\pi s}\hat{\varphi}}$.

Outside: $\oint \vec{H}\cdot d\vec{l} = H_\varphi 2\pi s = I_{encl} = 0 \Rightarrow \boxed{\vec{H} = 0}$.

Then $\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0\mu_r \vec{H} = \boxed{\begin{cases} \frac{\mu_0\mu_r I}{2\pi s}\hat{\varphi} & \text{inside} \\ 0 & \text{outside} \end{cases}}$.

and $\boxed{\vec{M} = \begin{cases} \frac{(\mu_r - 1) I}{2\pi s}\hat{\varphi} & \text{inside} \\ 0 & \text{outside} \end{cases}}$.

(b) $\vec{B} = \vec{\nabla}\times\vec{A} = \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\varphi}$. $\vec{J}$ points in the z direction, so set $A_s = 0$, $A_\varphi = 0$, and require $-\frac{\partial A_z}{\partial s} = \frac{\mu_0\mu_r I}{2\pi s} \Rightarrow$

$$A_z = -\frac{\mu_0\mu_r I}{2\pi}\int_{\text{zero}}^{s}\frac{ds}{s} = -\frac{\mu_0\mu_r I}{2\pi}\ln\left(\frac{s}{s_0}\right) \text{ for } s<R.$$

For $s \geq R$, $\vec{B} = 0$ everywhere, so we can set $\vec{A} = 0$ there.

Continuity then gives $\boxed{\vec{A} = \begin{cases} -\frac{\mu_0\mu_r I}{2\pi}\ln\left(\frac{s}{R}\right)\hat{z} & \text{for } s<R \\ 0 & \text{for } s \geq R \end{cases}}$

(c) $\vec{A} = A_z\hat{z} \Rightarrow \vec{\nabla}\cdot\vec{A} = \boxed{\frac{\partial A_z}{\partial z} = 0}$ since A_z is only a function of s.

Exam 1, Problem 4

(a) $\vec{p}_{EM} = \varepsilon_0 (\vec{E} \times \vec{B})$ requires both to be $\neq 0$. $\Rightarrow$ $\vec{p}_{EM} = 0$ everywhere except in the toroid. There $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} \approx \frac{q}{4\pi\varepsilon_0 a^2}\hat{s}$ and $\vec{B} = \frac{\mu_0 N I}{2\pi s}\hat{\phi} \approx \frac{\mu_0 N I}{2\pi a}\hat{\phi} \Rightarrow \vec{p}_{EM} = \cancel{\varepsilon_0}\frac{q}{4\pi\cancel{\varepsilon_0} a^2}\frac{\mu_0 N I}{2\pi a}\hat{s}\times\hat{\phi} \Rightarrow$

$\vec{p}_{EM} = \frac{\mu_0 N I q}{8\pi^2 a^3}\hat{z}$. Similarly, $\vec{\ell}_{EM} = 0$ everywhere except in the toroid where $\vec{\ell}_{EM} = \vec{r}\times\vec{p}_{EM} \approx a\hat{s}\times\frac{\mu_0 N I q}{8\pi^2 a^3}\hat{z} = -\frac{\mu_0 N I q}{8\pi^2 a^2}\hat{\phi}$

(b) $\vec{P}_{tot,EM} = \int \vec{p}_{EM}\, d\tau = \frac{\mu_0 N I q}{8\pi^2 a^3}\hat{z}(2\pi a h w) = \frac{\mu_0 N I q h w}{4\pi a^2}\hat{z}$

$\vec{L}_{tot,EM} = \int \vec{\ell}_{EM}\, d\tau = 0$ when $\hat{\phi}$ integrates around the toroid.

(c) $\vec{\mathcal{I}} = \int q\vec{E}\, dt$. The Biot-Savart Law for Faraday fields gives $\vec{E} = \frac{1}{4\pi}\int \frac{-\frac{\partial \vec{B}}{\partial t}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau$. This gives $\vec{\mathcal{I}} = \frac{-q}{4\pi}\int\frac{(\Delta\vec{B})\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau$

$= \frac{-q}{4\pi}\,\frac{\left(-\frac{\mu_0 N I}{\cancel{2\pi a}}\hat{\phi}\right)\times(-\hat{s})}{a^2}\,\cancel{2\pi a} h w = \frac{\mu_0 N I q h w}{4\pi a^2}(-\hat{\phi}\times\hat{s})$

$= \frac{\mu_0 N I q h w}{4\pi a^2}\hat{z}$, equal to $\vec{P}_{tot,EM}$ from part (b).