Physics 305 – Homework Set 8

Due submitted to eCampus no later than 5 pm on Thursday, Apr 9

Do the following nine problems from Griffiths:

10.20 (pg 462). Only worry about the fields to the right (x>x') of the particle.
For the special case v = constant, also reconcile your answer with Eq. 10.75.

10.27 (pg 463).

10.31 (pg 464).

11.2 (pg 472).

11.3 (pg 472).

11.4 (pg 473).

11.7 (pg 477).

11.8 (pg 482).

11.23 (pg 497).

Note: The answer key for this assignment will be posted shortly after it's due. However, it is unlikely to be fully graded before Exam 2.

Griffiths, Problem 10.20

For q moving on the x axis and $x > x'$, $\vec{u} = c\hat{\mathscr{r}} - \vec{v} = (c - v)\hat{x}$.
Also $\vec{a} = \dot{v}\hat{x} \Rightarrow \vec{u} \times \vec{a} = 0$, and the acceleration field drops out.

Meanwhile, $\vec{\mathscr{r}} = \mathscr{r}\hat{x}$, so $\vec{\mathscr{r}} \cdot \vec{u} = \mathscr{r}(c - v)$. Substituting into Eq. 10.72, we obtain:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{\mathscr{r}^{3}(c-v)^{3}} (c^2 - v^2)\cancel{(c-v)}\hat{x}$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{\mathscr{r}^2} \frac{\cancel{(c-v)}(c+v)}{\cancel{(c-v)}(c-v)}\hat{x} = \boxed{\frac{q}{4\pi\varepsilon_0} \frac{1}{\mathscr{r}^2} \frac{c+v}{c-v}\hat{x}}$$, as desired.

$\vec{E}$, $\vec{v}$, and $\hat{\mathscr{r}}$ all point along $\hat{x}$, so the cross product in Eq. 10.76 gives $\boxed{\vec{B} = 0}$, as desired.

<u>Reconcilation:</u>

WLOG, I can assume $x_{charge} = 0$ when $t = t_r \Rightarrow \mathscr{r} = x_{obs} = x$.

Then $\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{1}{x^2} \frac{c+v}{c-v}\hat{x}$. $\vec{R} = \vec{x} - \vec{v}(t - t_r)$

$$= \left(x - v\frac{x}{c}\right)\hat{x} = x\left(1 - \frac{v}{c}\right)\hat{x} \Rightarrow$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{\left(1-\frac{v}{c}\right)^2}{R^2} \frac{c+v}{c-v}\hat{R} = \frac{q}{4\pi\varepsilon_0} \frac{\hat{R}}{R^2} \left(1 - \frac{v}{c}\right)^{\cancel{2}} \frac{\left(1+\frac{v}{c}\right)}{\cancel{\left(1-\frac{v}{c}\right)}}$$

$$\boxed{= \frac{q}{4\pi\varepsilon_0} \frac{1 - \frac{v^2}{c^2}}{R^2}\hat{R}}$$, which is Eq. 10.75 for the present case $\sin\theta = 0$.

Griffiths, Problem 10.27

Let $f(\vec{r},t) = \left(c^2 t - \vec{r}\cdot\vec{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right)$, so

$V = \frac{qc}{4\pi\varepsilon_0}\frac{1}{\sqrt{f(\vec{r},t)}}$. Then $-\mu_0\varepsilon_0\frac{\partial V}{\partial t} = -\frac{\mu_0 q c}{4\pi}\left(\frac{-1}{2}\right)\frac{1}{\left(f(\vec{r},t)\right)^{3/2}}\frac{\partial f}{\partial t}$

$$= \frac{\mu_0 q c}{4\,\cancel{8}\pi}\frac{1}{\left[f(\vec{r},t)\right]^{3/2}}\left\{\cancel{2}\left(c^2 t - \vec{r}\cdot\vec{v}\right)c^2 + \left(c^2 - v^2\right)\left(-\cancel{2}c^2 t\right)\right\}$$

$$= \frac{\mu_0 q c}{4\pi}\frac{1}{\left[f(\vec{r},t)\right]^{3/2}}\left[c^2 v^2 t - c^2\vec{r}\cdot\vec{v}\right] = \boxed{\frac{\mu_0 q c^3}{4\pi}\frac{v^2 t - \vec{r}\cdot\vec{v}}{\left[f(\vec{r},t)\right]^{3/2}}}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\mu_0 q c}{4\pi}\frac{\left(\frac{-1}{2}\right)}{\left[f(\vec{r},t)\right]^{3/2}}\vec{v}\cdot\vec{\nabla}f$$

$$= \frac{-\mu_0 q c}{\cancel{8}\pi\;4}\frac{1}{\left[f(\vec{r},t)\right]^{3/2}}\vec{v}\cdot\left[\cancel{2}\left(c^2 t - \vec{r}\cdot\vec{v}\right)\left(-\vec{v}\right) + \left(c^2 - v^2\right)\left(\cancel{2}\vec{r}\right)\right]$$

$$= -\frac{\mu_0 q c}{4\pi}\frac{1}{\left[f(\vec{r},t)\right]^{3/2}}\left\{-c^2 t v^2 + \cancel{v^2\left(\vec{r}\cdot\vec{v}\right)} + c^2\left(\vec{r}\cdot\vec{v}\right) - \cancel{v^2\left(\vec{r}\cdot\vec{v}\right)}\right\}$$

$$\boxed{= \frac{\mu_0 q c^3}{4\pi}\frac{1}{\left[f(\vec{r},t)\right]^{3/2}}\left\{v^2 t - \vec{r}\cdot\vec{v}\right\}} \Rightarrow$$

$\boxed{\vec{\nabla}\cdot\vec{A} = -\mu_0\varepsilon_0\frac{\partial V}{\partial t}}$, as desired.

Griffiths, Problem 10.[redacted] 31

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\vec{E}\times\left[\frac{1}{c^2}(\vec{v}\times\vec{E})\right] = \varepsilon_0\left\{E^2\vec{v} - (\vec{E}\cdot\vec{v})\vec{E}\right\}.$$

Only the $\hat{x}$-component of $\vec{S}$ will survive the integration, so:

$\vec{S}\cdot\hat{x} = \varepsilon_0\left\{E^2 v - (E_x v)E_x\right\}$. If θ is the angle from the x-axis, then $\vec{S}\cdot\hat{x} = \varepsilon_0 E^2 v\left(1-\cos^2\theta\right) = \varepsilon_0 E^2 v \sin^2\theta \Rightarrow$

$$P = \int \vec{S}\cdot d\vec{a} = \frac{\cancel{\varepsilon_0} v\, q^2}{\cancel{16}_{8}\,\pi^{\cancel{2}}\varepsilon_0^{\cancel{2}}}\left(1-\frac{v^2}{c^2}\right)^2 \cancel{2\pi}\int_0^\infty s\,ds\,\frac{\sin^2\theta}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^3}\frac{1}{(s^2+a^2)^2}$$

$$= \frac{q^2 v}{8\pi\varepsilon_0}\left(1-\frac{v^2}{c^2}\right)^2\int_0^\infty s\,ds\,\frac{\frac{s^2}{s^2+a^2}}{\left(1-\frac{v^2}{c^2}\frac{s^2}{s^2+a^2}\right)^3}\frac{1}{(s^2+a^2)^2}$$

$$= \frac{q^2 v}{8\pi\varepsilon_0}\left(1-\frac{v^2}{c^2}\right)^2\int_0^\infty \frac{s^3\,ds}{\cancel{(s^2+a^2)^3}\left[1-\frac{v^2}{c^2}\frac{s^2}{s^2+a^2}\right]^3}\frac{\cancel{(s^2+a^2)^3}}{(s^2+a^2)^3}$$

$$= \frac{q^2 v}{8\pi\varepsilon_0}\left(1-\frac{v^2}{c^2}\right)^2\int_0^\infty \frac{s^3\,ds}{\left[a^2+s^2\left(1-\frac{v^2}{c^2}\right)\right]^3}$$

Let $s = \frac{a}{\sqrt{1-\frac{v^2}{c^2}}}\tan\theta \Rightarrow$

$$P = \frac{q^2 v}{8\pi\varepsilon_0}\cancel{\left(1-\frac{v^2}{c^2}\right)^2}\int_0^{\frac{\pi}{2}}\frac{\frac{a^3\tan^3\theta}{\cancel{\left(1-\frac{v^2}{c^2}\right)^{3/2}}}\frac{a}{\cancel{\left(1-\frac{v^2}{c^2}\right)^{1/2}}}\sec^2\theta\,d\theta}{\left[a^2+\frac{a^2}{\cancel{1-\frac{v^2}{c^2}}}\tan^2\theta\cancel{\left(1-\frac{v^2}{c^2}\right)}\right]^3}$$

$$= \frac{q^2 v}{8\pi\varepsilon_0}\frac{a^4}{a^{\cancel{6}\,2}}\int_0^{\frac{\pi}{2}}\frac{\tan^3\theta\sec^2\theta}{\sec^{\cancel{6}\,4}\theta}\,d\theta = \frac{q^2 v}{8\pi\varepsilon_0 a^2}\int_0^{\frac{\pi}{2}}\frac{\sin^3\theta}{\cancel{\cos^3\theta}}\cos^4\theta\,d\theta$$

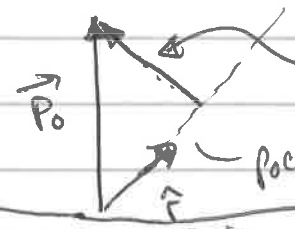
$$= \frac{q^2 v}{8\pi\varepsilon_0 a^2}\left[\frac{\sin^4\theta}{4}\right]_{\theta=0}^{\frac{\pi}{2}} = \boxed{\frac{q^2 v}{32\pi\varepsilon_0 a^2}}$$, as given in the text.

Griffiths, Problem 11.2

Eq. 11.14: $$\boxed{V = \frac{-\vec{p}_0 \cdot \hat{r}}{4\pi\varepsilon_0 c r}\,\omega \sin\left(\omega\left(t - \frac{r}{c}\right)\right)}$$

Eq. 11.17: $$\boxed{\vec{A} = -\frac{\mu_0}{4\pi r}\,\vec{p}_0\,\omega \sin\left(\omega\left(t - \frac{r}{c}\right)\right)}$$

Eq. 11.18:



$-p_0 \sin\theta\,\hat{\theta} = \vec{p}_0 - p_0\cos\theta\,\hat{r}$

$p_0\cos\theta\,\hat{r}$ $\qquad = \vec{p}_0 - (\vec{p}_0 \cdot \hat{r})\hat{r}$

Therefore, $$\boxed{\vec{E} = \frac{\mu_0}{4\pi r}\left[\vec{p}_0 - (\vec{p}_0\cdot\hat{r})\hat{r}\right]\omega^2 \cos\left(\omega\left(t - \frac{r}{c}\right)\right)}$$

It's straightforward to show that $\vec{p}_0 - (\vec{p}_0\cdot\hat{r})\hat{r} = (\hat{r}\times\vec{p}_0)\times\hat{r}$, so we can also write:

$$\boxed{\vec{E} = \frac{\mu_0}{4\pi r}\left((\hat{r}\times\vec{p}_0)\times\hat{r}\right)\omega^2 \cos\left(\omega\left(t - \frac{r}{c}\right)\right)}$$

Eq. 11.19: From the previous figure, we have $-p_0 \sin\theta\,\hat{\phi} = \hat{r}\times\vec{p}_0$

$$\Rightarrow \quad \boxed{\vec{B} = \frac{\mu_0}{4\pi r c}\left(\hat{r}\times\vec{p}_0\right)\omega^2 \cos\left(\omega\left(t - \frac{r}{c}\right)\right)}$$

Eq. 11.21: $p_0 \sin\theta = |\hat{r}\times\vec{p}_0| \Rightarrow \boxed{\langle \vec{S} \rangle = \frac{\mu_0 \omega^4}{32\pi^2 c}\,\frac{|\hat{r}\times\vec{p}_0|^2}{r^2}\,\hat{r}}$

Griffiths, Problem 11.3

$$P = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} = \frac{\mu_0 q_0^2 d^2 \omega^4}{12\pi c} = \langle I^2 R \rangle = \frac{1}{2} q_0^2 \omega^2 R$$

$$\Rightarrow R = \frac{\mu_0 d^2 \omega^2}{6\pi c}. \quad \omega = ck = \frac{2\pi c}{\lambda} \Rightarrow$$

$$R = \frac{\mu_0 d^2}{\cancel{6\pi c}\,3} \frac{\cancel{4\pi^2} c^{\cancel{2}}}{\lambda^2} = \frac{2\pi}{3} \mu_0 c \left(\frac{d}{\lambda}\right)^2$$

$$= \frac{2\pi}{3}\left(4\pi \times 10^{-7} \frac{N}{A^2}\right)\left(3.00 * 10^8 \frac{m}{s}\right)\left(\frac{d}{\lambda}\right)^2 = \boxed{790\,\Omega \left(\frac{d}{\lambda}\right)^2}$$

as desired.

The middle of the FM dial is $f \sim 100\text{ MHz} \Rightarrow \lambda = \dfrac{3.00 * 10^8 \frac{m}{s}}{\frac{10^8}{s}} = 3\,m$

Then $R = (790\,\Omega)\left(\dfrac{0.05\,m}{3\,m}\right)^2 \boxed{= 0.22\,\Omega}$

This is small, but perhaps not "negligible."

The middle of the AM dial is $f \sim 1\text{ MHz}$ (1000 kHz), so

$\lambda = 300\,m$ and $\boxed{R \sim 2.2 * 10^{-5}\,\Omega}$. This is definitely negligible.

11.4.1

Griffiths, Problem 11.4

From Problem 11.2 and superposition, $\boxed{\vec{E} = \vec{E}_1 + \vec{E}_2}$ where:

$$\boxed{\begin{aligned} \vec{E}_1 &= \frac{\mu_0 p_0 \omega^2}{4\pi r}\left[(\hat{r}\times\hat{x})\times\hat{r}\right]\cos\left(\omega\left(t-\frac{r}{c}\right)\right) \quad \text{and} \\ \vec{E}_2 &= \frac{\mu_0 p_0 \omega^2}{4\pi r}\left[(\hat{r}\times\hat{y})\times\hat{r}\right]\sin\left(\omega\left(t-\frac{r}{c}\right)\right) \end{aligned}}$$

$$\boxed{\vec{B} = \frac{1}{c}\hat{r}\times\vec{E} = \frac{1}{c}\left[\left(\hat{r}\times\vec{E}_1\right) + \left(\hat{r}\times\vec{E}_2\right)\right]}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}\left(\vec{E}\times\left(\hat{r}\times\vec{E}\right)\right) \boxed{= \frac{1}{\mu_0 c}E^2\hat{r}}$$ since $\vec{E}\perp\hat{r}$.

Time averages involves $\langle\cos^2\rangle = \frac{1}{2}$ for E_1^2, $\langle\sin^2\rangle = \frac{1}{2}$ for E_2^2 and $\langle\sin\cos\rangle = 0$ for the cross terms. Thus, for this particular relative phase, there is no interference in the intensity.

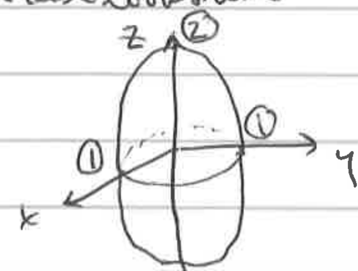
From Problem 11.2, we get:

$$\langle\vec{S}\rangle = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\,\frac{|\hat{r}\times\hat{x}|^2 + |\hat{r}\times\hat{y}|^2}{r^2}\,\hat{r}$$

$$|\hat{r}\times\hat{x}|^2 = \sin^2\theta_{\hat{r}\text{-}\hat{x}} = 1-\cos^2\theta_{\hat{r}\text{-}\hat{x}} = 1-\sin^2\theta\cos^2\varphi \quad \text{and}$$
$$|\hat{r}\times\hat{y}|^2 = \sin^2\theta_{\hat{r}\text{-}\hat{y}} = 1-\cos^2\theta_{\hat{r}\text{-}\hat{y}} = 1-\sin^2\theta\sin^2\varphi \quad \Rightarrow$$

$|\hat{r}\times\hat{x}|^2 + |\hat{r}\times\hat{y}|^2 = 2-\sin^2\theta = 1+\cos^2\theta$, and $\langle\vec{S}\rangle$ simplifies to:

$$\boxed{\langle\vec{S}\rangle = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\,\frac{1+\cos^2\theta}{r^2}\,\hat{r}}$$ Angular dist looks like



11.4.2

The total radiated power is:

From $\int d\varphi$

$$P = \int \langle \vec{S} \rangle r^2 d\Omega = \frac{\mu_0 p_0^2 \omega^4}{\overset{16}{\cancel{32}}\pi^{\cancel{2}} c} (\cancel{2}\cancel{\pi}) \int_{-1}^{1} (1 + x^2)\, dx$$

$$= \frac{\mu_0 p_0^2 \omega^4}{16\pi c} \left[x + \frac{x^3}{3} \right]_{x=-1}^{+1} = \frac{\mu_0 p_0^2 \omega^4}{\underset{2}{\cancel{16}}\pi c} \frac{\cancel{8}}{3} = \boxed{\frac{\mu_0 p_0^2 \omega^4}{6\pi c}}$$

This is double the result from Eq. 11.22, consistent with the observation above that the intensities of the $\hat{x}$-dipole and $\hat{y}$-dipole simply add without interference.

Note: You weren't asked about the polarization of the radiation. It's complicated! Along the $\hat{x}$- and $\hat{y}$-directions, only one wave contributes, and the light is linearly polarized. Along the $\hat{z}$-direction, both waves contribute equally, and the light is circularly polarized. In all other directions, the light is elliptically polarized.

Griffiths, Problem 11.7

To transform $q_e \to q_m = c q_e$, we need $\sin\alpha = -1$ in Eq 7.68 $\Rightarrow$ $\cos\alpha = 0$! Then: $\vec{E}' = -c\vec{B}$ and $\vec{B}' = \frac{1}{c}\vec{E}$. Taking $\vec{E}$ + $\vec{B}$ from Eqs. 11.18 + 11.19, we get:

$$\vec{E}' = -\cancel{c}\left[-\frac{\mu_0 p_0 \omega^2}{4\pi \cancel{c}}\left(\frac{\sin\theta}{r}\right)\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\hat{\varphi}\right]$$

$$\vec{B}' = \frac{1}{c}\left[-\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left(\omega\left(t-\frac{r}{c}\right)\right)\hat{\theta}\right]$$

Comparing to Eqs. 11.36 + 11.37, we find $\boxed{\vec{E}' = \frac{c}{c}\vec{E}_{\text{Mag Dipole}}}$ and $\boxed{\vec{B}' = \frac{c}{c}\vec{B}_{\text{Mag Dipole}}}$, consistent with $\boxed{m_0 = q_m d = c q_e d = c p_0}$.

The fields for Gilbert oscillating magnetic charges have exactly the same shape and magnitude as those of a conventional current loop magnetic dipole, so long as the magnetic dipole moments in the two cases are equal.

This is just what we should have expected. The only differences appear close to the dipole, as we discussed in Chapters 5+6.

Griffiths, Problem 11.8

(a) $p = Q_0 d e^{-\frac{t}{RC}} \Rightarrow \ddot{p} = \frac{Q_0 d}{R^2C^2} e^{-\frac{t}{RC}} \Rightarrow$ from Eq. 11.60, the radiated power is $P = \frac{\mu_0 Q_0^2 d^2}{6\pi c R^4 C^4} e^{-\frac{2t}{RC}} \Rightarrow$ the total radiated energy is

$$\int_0^\infty P\,dt = \frac{\mu_0 Q_0^2 d^2}{6\pi c R^4 C^4}\,\frac{RC}{2} = \frac{\mu_0 Q_0^2 d^2}{12\pi c R^3 C^3} = \frac{\mu_0 d^2}{6\pi c R^3 C^2}\,\frac{Q_0^2}{2C} \Rightarrow$$

the radiated fraction is $\boxed{\frac{\mu_0 d^2}{6\pi c R^3 C^2}}$

(b) Plugging in numbers, we get:

$$\frac{\left(\cancel{4}^{2}\cancel{\pi} * 10^{-7}\frac{N}{A^2}\right)\left(10^{-4}m\right)^2}{\cancel{6\pi}_{3}\left(3*10^8\frac{m}{s}\right)\left(1000\frac{V}{A}\right)^3\left(10^{-12}\frac{C}{V}\right)^2}$$

First, let's check the units: $\frac{\frac{N}{\cancel{A^2}}\,m^{\cancel{2}}}{\frac{\cancel{m}}{\cancel{s}}\,\frac{V^{\cancel{3}}}{\cancel{A^3}}\,\frac{C^{\cancel{2}}}{\cancel{V^2}}} = \frac{Nm}{VC}$

But $1V = 1\frac{J}{C}$, and $1\,Nm = 1\,J$, so this is $\boxed{\text{dimensionless}}$, as it should be. ✓

Numerically, we have:

$$\frac{\left(2*10^{-7}\right)\left(10^{-8}\right)}{\left(9*10^{8}\right)\left(10^{9}\right)\left(10^{-24}\right)} = \frac{2}{9}*10^{-15-17+24} = \frac{2}{9}*10^{-8}$$

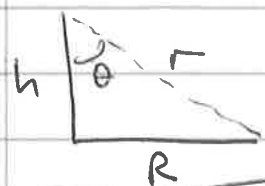
$$\boxed{\approx 2.2*10^{-9}}$$

This is <u>definitely</u> negligible.

Griffiths, Problem 11.23

(a) From Eq. 11.40, $\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$, while Eq. 11.39 ~~gt~~ gives

$$\langle \vec{S} \rangle = \frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r} = \frac{\cancel{12}^{\,3}}{\cancel{32}_{\,8}\,\pi} P \frac{\sin^2\theta}{r^2}\hat{r}$$



We have $\sin\theta = \frac{R}{r} = \frac{R}{\sqrt{R^2+h^2}}$ ⇒ the radiation intensity magnitude at ground level is:

$$\boxed{I = \frac{3}{8\pi} P \frac{R^2}{(R^2+h^2)^2}}$$

(b) $$\frac{dI}{dR} = \frac{(R^2+h^2)^2[2R] - R^2[2(R^2+h^2)2R]}{(R^2+h^2)^4} = 0$$

$$\Rightarrow \left[(R^2+h^2) - 2R^2\right]\cancel{(2R)}\cancel{(R^2+h^2)} = 0 \Rightarrow h^2 - R^2 = 0 \Rightarrow$$

The engineer should have measured a $\boxed{\text{distance } h}$ from the base of the tower. At this point, we find:

$$I = \frac{3}{8\pi} P \frac{h^2}{(2h^2)^2} \boxed{= \frac{3}{32\pi}\frac{P}{h^2}}$$

(c) The power level for KRUD is $\frac{3}{32\pi}\frac{35{,}000\,W}{(200\,m)^2} = \frac{0.026\,W}{m^2}$

$$\frac{0.026\,W}{\cancel{m^2}}\;\frac{1\,\cancel{m^2}}{10^4\,cm^2} = 2.6\,\frac{\mu W}{cm^2}$$. Looks like KRUD is legal. (I wasn't expecting that!)