Physics 305 – Homework Set 7

Due submitted to eCampus no later than 5 pm on Wednesday, Apr 1

Do the following five problems from Griffiths:

10.4 (pg 440).

10.6 (pg 442).

10.10 (pg 448).

10.11 (pg 448).

10.14 (pg 450-51).

Griffiths, Problem 10.4

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t}\left(A_0 \sin(kx - \omega t)\,\hat{y}\right) = (-A_0)(-\omega)\cos(k\cancel{x} - \omega t)\,\hat{y}$$

$$\Rightarrow \boxed{\vec{E} = A_0 \omega \cos(kx - \omega t)\,\hat{y}}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \boxed{A_0 k \cos(kx - \omega t)\,\hat{z}}$$

If $\boxed{\omega = ck}$, this will be an EM wave propagating in the $+\hat{x}$-direction and polarized in the $\hat{y}$-direction. (If $\omega = -ck$, it will propagate in $-x$.)

Check: $\vec{E} = E(x,t)\,\hat{y}$ and $\vec{B} = B(x,t)\,\hat{z}$, so $\vec{\nabla}\cdot\vec{E} = 0$ + $\vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla} \times \vec{E} = -A_0 \omega k \sin(kx - \omega t)\,\hat{z}$$ Match

$$-\frac{\partial \vec{B}}{\partial t} = -A_0 k(-\omega)(-\sin(kx - \omega t))\,\hat{z} = -A_0 \omega k \sin(kx - \omega t)\,\hat{z}.$$

$$\vec{\nabla} \times \vec{B} = (-A_0 k)\left[-k \sin(kx - \omega t)\right]\hat{y} = A_0 k^2 \sin(kx - \omega t)\,\hat{y}$$

$$\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = (\mu_0 \varepsilon_0 A_0 \omega)(\omega \sin(kx - \omega t))\,\hat{y} = \mu_0 \varepsilon_0 A_0 \omega^2 \sin(kx - \omega t)\,\hat{y}.$$

These match if $\omega^2 = \frac{1}{\mu_0 \varepsilon_0} k^2 = c^2 k^2 \Rightarrow \boxed{\omega = \pm ck}.$

ω is always ≥ 0; the sign of k determines the propagation direction.

Griffiths, Problem 10.6

Assume we have a solution $\vec{A}, V$. Then we know:

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$$
$$V' = V - \frac{\partial\lambda}{\partial t}$$ will also work, for any function $\lambda(\vec{r},t)$. Then

$$\vec{\nabla}\cdot\vec{A}' + \mu_0\varepsilon_0\frac{\partial V'}{\partial t} = \vec{\nabla}\cdot\left(\vec{A}+\vec{\nabla}\lambda\right) + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(V - \frac{\partial\lambda}{\partial t}\right)$$

$$= \vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t} + \nabla^2\lambda - \mu_0\varepsilon_0\frac{\partial^2\lambda}{\partial t^2}$$

$\vec{A}', V'$ will obey the Lorenz gauge condition if this $= 0 \Rightarrow$

$$\nabla^2\lambda - \mu_0\varepsilon_0\frac{\partial^2\lambda}{\partial t^2} = -\left(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)$$

This is the inhomogeneous wave equation. If we solve it for λ, we'll have an $\vec{A}', V'$ than works.

If we choose $\lambda = \int_0^t V\,dt'$, the $\frac{\partial\lambda}{\partial t} = V \Rightarrow V' = V - V = 0$.

Thus, $\vec{A}', 0$ will give $\vec{E}, \vec{B}$, and we can choose $V = 0$.

If $\vec{A} = 0$, then $\vec{B} = \vec{\nabla}\times\vec{A} = 0$. Thus, $\vec{A} = 0$ is <u>not possible</u> in general.

Griffiths, Problem 10.10

$$\vec{\nabla}\cdot\left(\frac{\vec{J}}{\imath}\right) = \frac{1}{\imath}\left(\vec{\nabla}\cdot\vec{J}\right) + \vec{J}\cdot\left(\vec{\nabla}\left(\frac{1}{\imath}\right)\right)$$

$$\vec{\nabla}'\cdot\left(\frac{\vec{J}}{\imath}\right) = \frac{1}{\imath}\left(\vec{\nabla}'\cdot\vec{J}\right) + \vec{J}\cdot\vec{\nabla}'\left(\frac{1}{\imath}\right)$$

But $\vec{\imath} = \vec{r} - \vec{r}' \Rightarrow \vec{\nabla}'\left(\frac{1}{\imath}\right) = -\vec{\nabla}\left(\frac{1}{\imath}\right) \Rightarrow$

$$\vec{\nabla}\cdot\left(\frac{\vec{J}}{\imath}\right) + \vec{\nabla}'\cdot\left(\frac{\vec{J}}{\imath}\right) = \frac{1}{\imath}\left(\vec{\nabla}\cdot\vec{J}\right) + \frac{1}{\imath}\left(\vec{\nabla}'\cdot\vec{J}\right)$$. Rearranging gives:

① $$\boxed{\vec{\nabla}\cdot\left(\frac{\vec{J}}{\imath}\right) = \frac{1}{\imath}\left(\vec{\nabla}\cdot\vec{J}\right) + \frac{1}{\imath}\left(\vec{\nabla}'\cdot\vec{J}\right) - \vec{\nabla}'\cdot\left(\frac{\vec{J}}{\imath}\right)}$$, as given.

$\vec{J} = \vec{J}\left(\vec{r}', t - \frac{\imath}{c}\right) \Rightarrow$ ② $\boxed{\vec{\nabla}\cdot\vec{J} = \dot{\vec{J}}\cdot\left(\frac{-\vec{\nabla}(\imath)}{c}\right)}$ (Griffiths definition of "dot")

$$\vec{\nabla}'\cdot\vec{J} = \vec{\nabla}'\cdot\vec{J}\Big|_{\text{const } t_r} + \dot{\vec{J}}\cdot\left(\frac{-\vec{\nabla}'\imath}{c}\right)$$

But, by the continuity equation, $\vec{\nabla}'\cdot\vec{J}(\vec{r}', t_r = \text{const}) = -\dot{\rho}(\vec{r}', t_r)$

③ $\Rightarrow \boxed{\vec{\nabla}'\cdot\vec{J} = -\dot{\rho} - \frac{1}{c}\dot{\vec{J}}\cdot\vec{\nabla}'(\imath)}$. Plugging everything in ① gives:

$$\vec{\nabla}\cdot\left(\frac{\vec{J}}{\imath}\right) = -\frac{1}{\imath c}\dot{\vec{J}}\cdot\vec{\nabla}(\imath) - \frac{1}{\imath c}\dot{\vec{J}}\cdot\vec{\nabla}'(\imath) - \frac{\dot{\rho}}{\imath} - \vec{\nabla}'\cdot\left(\frac{\vec{J}}{\imath}\right)$$

$\vec{\nabla}'(\imath) = -\vec{\nabla}(\imath)$, so the first two terms cancel. Then:

$$\vec{\nabla}\cdot\vec{A} = \frac{\mu_0}{4\pi}\int\vec{\nabla}\cdot\left(\frac{\vec{J}}{\imath}\right)d\tau' = \frac{\mu_0}{4\pi}\int\left(-\frac{\dot{\rho}}{\imath} - \vec{\nabla}'\cdot\left(\frac{\vec{J}}{\imath}\right)\right)d\tau'$$

$$= -\frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\left(\int\frac{\rho}{\imath}d\tau'\right) - \cancel{\frac{\mu_0}{4\pi}\oint\frac{\vec{J}}{\imath}\cdot d\vec{a}'}$$ Goes to zero if no $\vec{J}$ at large sphere $\Rightarrow$

$$\boxed{\vec{\nabla}\cdot\vec{A} = -\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\frac{1}{4\pi\varepsilon_0}\int\frac{\rho}{\imath}d\tau'\right) = -\mu_0\varepsilon_0\frac{\partial V}{\partial t}}$$, as desired.

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Griffiths, Problem 10.11

(a) This repeats the calculation in Ex. 10.2, but with $I(t_r) = I_0$ replaced by $I(t_r) = I\left(t - \frac{r}{c}\right) = k\left(t - \frac{\sqrt{s^2+z^2}}{c}\right) \Rightarrow$

$$\vec{A}(\vec{s},t) = \frac{\mu_0 k}{2\pi}\hat{z}\int_0^{\sqrt{(ct)^2-s^2}} \frac{t - \frac{\sqrt{s^2+z^2}}{c}}{\sqrt{s^2+z^2}}\,dz$$

$$= \frac{\mu_0 k t}{2\pi}\hat{z}\int_0^{\sqrt{(ct)^2-s^2}} \frac{dz}{\sqrt{s^2+z^2}} - \frac{\mu_0 k}{2\pi c}\hat{z}\int_0^{\sqrt{(ct)^2-s^2}} \frac{\cancel{\sqrt{s^2+z^2}}}{\cancel{\sqrt{s^2+z^2}}}\,dz$$

$$= \underbrace{\frac{\mu_0 k t}{2\pi}\hat{z}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)}_{\text{This is }\vec{A}\text{ from Ex. 10.2 with }I_0 \to kt} - \frac{\mu_0 k}{2\pi c}\hat{z}\left(\sqrt{(ct)^2-s^2}\right)$$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = \Bigg[\underbrace{-\frac{\mu_0 k t c}{2\pi\sqrt{(ct)^2-s^2}} - \frac{\mu_0 k}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)}_{\text{From Ex 10.2}}$$

$$+ \frac{\mu_0 k}{2\pi \cancel{c}}\frac{1}{\cancel{2}}\frac{\cancel{2}c^{\cancel{2}}t}{\sqrt{(ct)^2-s^2}}\Bigg]\hat{z} \; \boxed{= -\frac{\mu_0 k}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)\hat{z}}$$

for $s < ct$ and zero for $s > ct$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = \Bigg[\underbrace{\frac{\mu_0 k t}{2\pi s}\frac{ct}{\sqrt{(ct)^2-s^2}}}_{\text{From Ex 10.2}} + \frac{\mu_0 k}{2\pi c}\frac{1}{\cancel{2}}\frac{(-\cancel{2}s)}{\sqrt{(ct)^2-s^2}}\Bigg]\hat{\phi}$$

$$= \frac{\mu_0 k}{2\pi c s}\frac{(ct)^2-s^2}{\sqrt{(ct)^2-s^2}}\hat{\phi} \; \boxed{= \frac{\mu_0 k}{2\pi c s}\sqrt{(ct)^2-s^2}\,\hat{\phi}}$$

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(b) $\vec{A}(s,t) = \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{+\infty} \frac{q_0\,\delta(t_r)}{\mathscr{r}}\,dz$ with $t_r = t - \frac{\mathscr{r}}{c} = t - \frac{\sqrt{z^2+s^2}}{c}$

$$\delta(t_r(z)) = \sum_{\text{zeros}} \frac{1}{\left|\frac{dt_r}{dz}\right|_{@\text{ZERO}}}\,\delta(z-z_0).$$

$\frac{dt_r}{dz} = -\frac{1}{c}\,\frac{1}{\cancel{2}}\,\frac{\cancel{2}z}{\sqrt{z^2+s^2}} = -\frac{1}{c}\,\frac{z}{\sqrt{z^2+s^2}}$. The zeros occur when $t_r = 0 \Rightarrow$ $(ct)^2 = z^2+s^2 \Rightarrow \left|\frac{dt_r}{dz}\right|_{\text{ZERO}} = \frac{1}{c}\,\frac{\sqrt{(ct)^2-s^2}}{ct} = \frac{\sqrt{(ct)^2-s^2}}{c^2 t}$. Thus,

$$\vec{A} = \frac{\mu_0 q_0}{\underset{2}{\cancel{4\pi}}}\,\hat{z}\,\underbrace{(\cancel{2})}_{\text{Two zeros}}\,\frac{c^2 t}{\sqrt{(ct)^2-s^2}}\,\underbrace{\frac{1}{(\cancel{c}\cancel{t})}}_{\mathscr{r}=ct \text{ at } t_r=0} \boxed{= \frac{\mu_0 q_0 c}{2\pi\sqrt{(ct)^2-s^2}}\,\hat{z}}$$ for $ct > s$, zero for $ct < s$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} = \frac{-\mu_0 q_0 c}{2\pi}\,\hat{z}\left(-\frac{1}{\cancel{2}}\,\frac{\cancel{2}c^2 t}{((ct)^2-s^2)^{3/2}}\right) \boxed{= \frac{\mu_0 q_0 c^3}{2\pi}\,\frac{t}{[(ct)^2-s^2]^{3/2}}\,\hat{z}}$$

for $ct > s$ and zero for $ct < s$.

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial s}\,\hat{\phi} = \frac{-\mu_0 q_0 c}{2\pi}\left(+\frac{1}{\cancel{2}}\,\frac{+\cancel{2}s}{[(ct)^2-s^2]^{3/2}}\right)\hat{\phi}$$

$$\boxed{= \frac{-\mu_0 q_0 c}{2\pi}\,\frac{s}{[(ct)^2-s^2]^{3/2}}\,\hat{\phi}}$$ for $ct > s$ and zero for $ct < s$.

Griffiths, Problem 10.14

From Eq. 10.38:

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r}',t_r)}{\eta^2} + \frac{\dot{\vec{J}}(\vec{r}',t_r)}{c\eta}\right]\times\hat{\eta}\,d\tau'$$

$$= \frac{\mu_0}{4\pi}\int\frac{1}{\eta^2}\left[\vec{J}(\vec{r}',t_r) + \frac{\eta}{c}\dot{\vec{J}}(\vec{r}',t_r)\right]\times\hat{\eta}\,d\tau'$$

But $t_r = t - \frac{\eta}{c} \Rightarrow \frac{\eta}{c} = t - t_r$. Then substituting the Taylor expansion from the problem, we find:

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{1}{\eta^2}\left[\vec{J}(\vec{r}',t) + (t_r - t)\dot{\vec{J}}(\vec{r}',t) + (t - t_r)\dot{\vec{J}}(\vec{r}',t_r)\right]\times\hat{\eta}\,d\tau'$$

If we can ignore all higher derivatives, then $\vec{J}$ is a linear function of time and $\dot{\vec{J}}(\vec{r}',t_r) = \dot{\vec{J}}(\vec{r}',t)$. In this limit, the second + third terms cancel, giving:

$$\boxed{\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}',t)\times\hat{\eta}}{\eta^2}\,d\tau'}$$, as desired.