Physics 305 – Homework Set 6

~~Due in class on Wednesday, Mar. 18~~

Due date delayed until Wednesday, Mar. 25, because of the COVID-19 class cancelations

Do the following four problems from Griffiths:

9.28 (pg 430).

9.30 (pg 430).

9.31 (pg 431).

9.40 (pg 435).

Griffiths, Problem 9.28

A TE_{00} mode would have:

$$B_z(x,y,t) = B_0 \cos(0x)\cos(0y)\cos(kz-\omega t)$$
$$= B_0 \cos(kz-\omega t).$$

At a fixed z, & if we choose t=0 appropriately, this is just

$$B_z(x,y,t) = B_0 \cos(\omega t) \Rightarrow$$

$$\Phi = B_0 ab \cos(\omega t) \Rightarrow \mathcal{E} = -\frac{d\Phi}{dt} = B_0 ab \sin(\omega t)\,\omega.$$

But $\vec{E}_{\parallel} = 0$ at the surface of the conductor $\Rightarrow \oint \vec{E}\cdot d\vec{l} = 0 = \mathcal{E} \Rightarrow$

$\boxed{B_0 = 0}$ as desired.

$\Rightarrow$ A TE_{00} mode would be a TEM mode, which can't exist in a (single) hollow conductor.

9.30.1

Griffiths, Problem 9.30

For the TE_{mn} mode, $E_z = 0$ and $B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$

Plugging into Eq. 9.180, we find:

$$E_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\,\omega\left[B_0 \cos\left(\frac{m\pi x}{a}\right)\left(-\frac{n\pi}{b}\sin\left(\frac{n\pi y}{b}\right)\right)\right]$$

$$E_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}(-\omega)\left[B_0\left(-\frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\right)\cos\left(\frac{n\pi y}{b}\right)\right]$$

$$B_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\,k\left[B_0\left(-\frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\right)\cos\left(\frac{n\pi y}{b}\right)\right]$$

$$B_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\,k\left[B_0 \cos\left(\frac{m\pi x}{a}\right)\left(-\frac{n\pi}{b}\sin\left(\frac{n\pi y}{b}\right)\right)\right]$$

The time-average electric energy density is:

$$\langle u_E\rangle = \frac{\varepsilon_0}{4}\left(|E_x|^2 + |E_y|^2\right) = \frac{\varepsilon_0}{4}\frac{B_0^2\omega^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2}\left(\frac{n^2\pi^2}{b^2}\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) + \frac{m^2\pi^2}{a^2}\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)\right)$$

If we integrate this over $x+y$, we get:

$$\int\langle u_E\rangle\,da = \frac{\varepsilon_0}{4}\frac{B_0^2\omega^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2}\left[\frac{n^2\pi^2}{b^2}\left(\frac{a}{2}(1+\delta_{m0})\frac{b}{2}\right) + \frac{m^2\pi^2}{a^2}\left(\frac{a}{2}\frac{b}{2}(1+\delta_{n0})\right)\right]$$

For future convenience, substitute $\varepsilon_0 = \frac{1}{\mu_0 c^2}$, $n^2 = n^2(1+\delta_{n0})$, $m^2 = m^2(1+\delta_{m0})$ to get:

$$\int\langle u_E\rangle\,da = \frac{ab\pi^2}{16}\frac{B_0^2\left(\frac{\omega}{c}\right)^2}{\mu_0\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2}(1+\delta_{n0})(1+\delta_{m0})\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

Likewise, $\langle u_B\rangle = \frac{1}{4\mu_0}\left(|B_x|^2 + |B_y|^2 + |B_z|^2\right)$, and if we integrate it over $x+y$, we find:

$$\int\langle u_B\rangle\,da = \frac{1}{4\mu_0}\frac{B_0^2k^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2}\left[\frac{m^2\pi^2}{a^2}\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)(1+\delta_{n0}) + \frac{n^2\pi^2}{b^2}\left(\frac{a}{2}(1+\delta_{m0})\frac{b}{2}\right)\right]$$

$$+ \frac{1}{4\mu_0}B_0^2\left(\frac{a}{2}(1+\delta_{m0})\right)\left(\frac{b}{2}(1+\delta_{n0})\right)$$

$$= \frac{ab}{16\mu_0} B_0^2 (1+\delta_{m0})(1+\delta_{n0}) \left[1 + \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right) \frac{k^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2} \right]$$

Adding, we get:

$$\boxed{\int \langle u \rangle da = \frac{\pi^2 ab}{16} \frac{B_0^2 (1+\delta_{m0})(1+\delta_{n0})}{\mu_0 \left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2} \left[\left(\left(\frac{\omega}{c}\right)^2 + k^2 \right)\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) + \frac{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2}{\pi^2} \right]}$$

$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$. B_z is $90°$ out of phase with $\vec{E}$, so it makes no contribution to the time average. The time-averaged intensity is:

$$I = \frac{1}{2\mu_0} \left(E_x B_y^* - E_y B_x^* \right)$$

$$= \frac{1}{2\mu_0} \frac{\omega k B_0^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2} \left\{ \frac{n^2\pi^2}{b^2} \cos^2\left(\frac{m\pi x}{a}\right) \sin^2\left(\frac{n\pi y}{b}\right) + \frac{m^2\pi^2}{a^2} \sin^2\left(\frac{m\pi x}{a}\right) \cos^2\left(\frac{n\pi y}{b}\right) \right\}$$

Integrating over x & y, we get:

$$\boxed{\int I \, da = \frac{1}{2\mu_0} \frac{\omega k B_0^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2} \frac{\pi^2 ab}{4} (1+\delta_{m0})(1+\delta_{n0}) \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}$$

The speed of energy transport is given by:

$$v = \frac{\int I \, da}{\int \langle u \rangle da} = \frac{2\omega k \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}{\left(\left(\frac{\omega}{c}\right)^2 + k^2 \right)\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) + \frac{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2}{\pi^2}}$$

Eq. 9.187 gives $\frac{\left(\frac{\omega}{c}\right)^2 - k^2}{\pi^2} = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \Rightarrow$

$$v = \frac{2\omega k \cancel{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}}{\left(\left(\frac{\omega}{c}\right)^2 + \cancel{k^2} \right)\cancel{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} + \left(\left(\frac{\omega}{c}\right)^2 - \cancel{k^2} \right)\cancel{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}} = \frac{\cancel{2}\omega k}{\cancel{2}\left(\frac{\omega}{c}\right)^2} = c^2 \frac{k}{\omega}$$

(Eq 9.190) $\Rightarrow$ $c^2 \frac{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}}{c} = \boxed{c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}} = v_g$, as desired.

Griffiths, Problem 9.31

E_z must obey $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]E_z=0$, while $B_z=0$

Separation of variables gives:
$E_z=\left(A\cos(k_x x)+B\sin(k_x x)\right)\left(C\cos(k_y y)+D\sin(k_y y)\right)$.

$\vec{E}_{\parallel}$ must be continuous at the edges $\Rightarrow$ must have $E_z=0$ at $x=0$ and $x=a$, and at $y=0$ and $y=b$. $\Rightarrow$ $A+C=0$, $k_x=\frac{m\pi}{a}$, $k_y=\frac{n\pi}{b}$. Also, must have both $m,n\neq 0$ for $E_z\neq 0 \Rightarrow$

$\boxed{E_z=E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}$, with m,n positive integers.

Does this obey the boundary conditions for E_x, E_y, B_x, B_y?

$E_x=\frac{ik}{(\frac{\omega}{c})^2-k^2}\frac{\partial E_z}{\partial x} \propto \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$. This is zero at $y=0$ and $y=b$, as it must be. ✓

$B_y \propto E_x$, so it is also zero at $y=0$ + $y=b$, as required for $\vec{B}_\perp=0$ at the surfaces. ✓

$E_y=\frac{ik}{(\frac{\omega}{c})^2-k^2}\frac{\partial E_z}{\partial y} \propto \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$. This is zero at $x=0$ and $x=a$, as it must be. ✓

$B_x \propto E_y$, so it is also zero at $x=0$ and $x=a$, as needed. ✓

<u>Works!</u> The above E_z is the most general TM wave.

Substituting back into the E_z equation, we find:

$\frac{\omega^2}{c^2} = k^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$. The cut-off forequencies are given by:

$\boxed{\omega_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}}$; same as for TE waves, except that

mn = 1,0 and 0,1 are no longer allowed.

The wave and group velocities are unchanged:

$\boxed{v = \frac{\omega}{k} = \frac{c}{\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2}}}$; $\boxed{v_g = \frac{d\omega}{dk} = c\sqrt{1-\left(\frac{\omega_{mn}}{\omega}\right)^2}}$

The lowest cut-off frequencies are ω_{10} for TE and

ω_{11} for TM $\Rightarrow$ Ratio $= \frac{\omega_{11}}{\omega_{10}} = \frac{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}{\sqrt{\frac{1}{a^2}}} = \boxed{\sqrt{1+\frac{a^2}{b^2}}}$

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Griffiths, Problem 9.40

<u>TE mode</u>: Write $\vec{B}$ as the supn of TE waves propagating in the $\pm z$ directions: $\vec{B}_{tot} = \tilde{\vec{B}}_+(x,y)e^{i(k_{z_+}z - \omega_+ t)} + \tilde{\vec{B}}_-(x,y)e^{i(-k_{z_-}z - \omega_- t)}$

(I'll drop ~'s)

We know $B_z(z=0)=0$ and $B_z(z=d)=0$, for all x, y, t. Thus, $\omega_+ = \omega_- = \omega$ and $B_{z_+}(x,y) + B_{z_-}(x,y) = 0 \Rightarrow$

$$B_z(x,y,z,t) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\left[e^{ik_{z_+}z} - e^{-ik_{z_-}z}\right]e^{-i\omega t}$$

Let $\boxed{k_x = \frac{m\pi}{a},\ k_y = \frac{n\pi}{b}}$. Then from 9.183, $k_{z_+} = k_{z_-} = k_z$. Redefine B_0 to:

$$\boxed{B_z(x,y,z,t) = B_0 \cos(k_x x)\cos(k_y y)\sin(k_z z)e^{-i\omega t}}$$

We must have $B_z(z=d)=0 \Rightarrow \boxed{k_z = \frac{l\pi}{d}}$, and this is the most general TE mode form for B_z.

To get E_x, E_y, B_x, B_y, plug the $+z$ and $-z$ waves into 9.180 separately. Dropping the $e^{-i\omega t}$ terms:

$$E_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\left[\frac{B_0\omega}{2i}\right](\cos k_x x)\left[-k_y \sin k_y y\right]\left(e^{ik_z z} - e^{-ik_z z}\right) \Rightarrow$$

$$\boxed{E_x = \frac{-iB_0\omega k_y}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\cos(k_x x)\sin(k_y y)\sin(k_z z)}$$

Note that this obeys our b.c.'s $E_x(z=0) = E_x(z=d) = 0$.

$$E_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\left[\frac{-B_0\omega}{2i}\right](-k_x \sin k_x x)(\cos(k_y y))\left(e^{ik_z z} - e^{-ik_z z}\right) \Rightarrow$$

$$\boxed{E_y = \frac{iB_0\omega k_x}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\sin(k_x x)\cos(k_y y)\sin(k_z z)}$$

Again, this obeys $E_y(z=0) = E_y(z=d) = 0$.

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For B_x and B_y, we need to flip the sign of k in 9.180 (iii) + (iv) for the $-z$ wave:

$$B_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\left(\frac{B_0 k_z}{2i}\right)(-k_x \sin k_x x)(\cos k_y y)\left(e^{ik_z z} + e^{-ik_z z}\right) \Rightarrow$$

$$\boxed{B_x = \frac{-B_0 k_z k_x}{\left(\frac{\omega}{c}\right)^2 - k_z^2}(\sin k_x x)(\cos k_y y)\cos(k_z z)}$$

$$B_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\left(\frac{B_0 k_z}{2i}\right)(\cos k_x x)(-k_y \sin k_y y)\left(e^{ik_z z} + e^{-ik_z z}\right) \Rightarrow$$

$$\boxed{B_y = \frac{-B_0 k_z k_y}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\cos(k_x x)\sin(k_y y)\cos(k_z z)}$$

and $k_x^2 + k_y^2 + k_z^2 = \boxed{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2 = \left(\frac{\omega}{c}\right)^2}$, as desired.

<u>TM mode:</u>

Here we use the same approach, starting from $E_z \propto \sin(k_x x)\sin(k_y y)$. However, the $E_z + B_z$ boundary conditions at $z = 0 + d$ are trivial, so we need to look to the x- and y-components to constrain the z-dependence of E_z beyond $\frac{e^{ik_z z} + \alpha e^{-ik_z z}}{2}$. ~~for necessary~~

So start with:

$$E_z = E_0 \sin(k_x x)\sin(k_y y)\frac{e^{ik_z z} + \alpha e^{-ik_z z}}{2} \Rightarrow$$

$$E_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\left(\frac{E_0 k_z}{2}\right)(k_x \cos k_x x)(\sin k_y y)\left(e^{ik_z z} - \alpha e^{-ik_z z}\right)$$

$E_x(z=0) = 0 \Rightarrow \alpha = 1$. $E_x(z=d) = 0 \Rightarrow \boxed{k_z = \frac{l\pi}{d}}$, so:

$$\boxed{E_z = E_0 \sin(k_x x)\sin(k_y y)\cos(k_z z)}$$

$$\boxed{E_x = \frac{-E_0 k_z k_x}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\cos(k_x x)\sin(k_y y)\sin(k_z z)}$$

obeys all E_x boundary conditions by construction.

$$E_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\left(\frac{E_0 k_z}{2}\right)(\sin k_x x)(k_y \cos k_y y)\left(e^{ik_z z} - e^{-ik_z z}\right) \Rightarrow$$

$$\boxed{E_y = \frac{-E_0 k_z k_y}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\sin(k_x x)\cos(k_y y)\sin(k_z z)}$$

This obeys $E_y(z=0) = E_y(z=d) = 0$. ✓

$$B_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\left(\frac{-E_0 \omega}{c^2}\right)(\sin k_x x)(k_y \cos k_y y)\left(\frac{e^{ik_z z} + e^{-ik_z z}}{2}\right) \Rightarrow$$

$$\boxed{B_x = \frac{-iE_0 \omega k_y}{c^2\left(\left(\frac{\omega}{c}\right)^2 - k_z^2\right)}\sin(k_x x)\cos(k_y y)\cos(k_z z)}$$

$$B_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k_z^2}\left(\frac{E_0 \omega}{c^2}\right)(k_x \cos k_x x)\sin(k_y y)\frac{e^{ik_z z} + e^{-ik_z z}}{2} \Rightarrow$$

$$\boxed{B_y = \frac{iE_0 \omega k_x}{c^2\left[\left(\frac{\omega}{c}\right)^2 - k_z^2\right]}\cos(k_x x)\sin(k_y y)\cos(k_z z)}$$

And once again, $k_x^2 + k_y^2 + k_z^2 = \boxed{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2 = \left(\frac{\omega}{c}\right)^2}$,

as desired.

Note that the x, y, z, t dependences of E_x, E_y, B_x, B_y are the same for TE and TM modes.