Physics 305 – Homework Set 5

Due in my mailbox before 4 pm on Thursday, Mar. 5

Do the following five problems from Griffiths:

9.19 (pg 415).
Skip part (a). In parts (b) and (c), assume that $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ for both metals.

9.21 (pg 416).

9.25 (pg 424).

9.26 (pg 424)
For part (b), also make graphs of the phase velocity, v_p/c, and absorption coefficient, α, over the same range of "x".
Note: This is another problem to give you a sense of the numbers.

9.39 (pg 433-35).

Griffiths, Problem 9.19

(b) From Eq. 9.128, $d = \frac{1}{K}$.

From Eq. 9.126, $K = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}$.

Table 7.1 gives $\rho = \frac{1}{\sigma} = 1.59 * 10^{-8}\ \Omega\cdot m$. Thus, taking $\varepsilon = \varepsilon_0$ & $\mu = \mu_0$,

we have $\frac{\sigma}{\varepsilon\omega} = \frac{1}{(1.59*10^{-8}\ \Omega\cdot m)\left(8.85*10^{-12}\frac{C^2}{Nm^2}\right)\left(2\pi * \frac{10^{10}}{s}\right)}$

$= 1.13 * 10^{8}$ (a pure number — the units really work!)

$\Rightarrow K \simeq 2\pi * \frac{10^{10}}{\cancel{s}}\ \frac{1}{\sqrt{2}}\ \frac{1}{3.00*10^{8}\frac{m}{\cancel{s}}}\left(1.13*10^{8}\right)^{1/2} = \frac{1.58*10^{6}}{m} \Rightarrow$

The skin depth is $0.64 * 10^{-6}\ m = \boxed{0.64\ \mu m}$.

Make the silver coating a few times this, and you'll be fine.

(c) Once again, $\frac{\sigma}{\varepsilon\omega} \gg 1$, so $k \approx \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\mu\omega\sigma}{2}}$

$= \sqrt{\left(4\pi*10^{-7}\frac{N}{A^2}\right)\left(2\pi*\frac{10^6}{s}\right)\frac{1}{(1.68*10^{-8}\ \Omega\cdot m)}\ \frac{1}{2}} = \frac{1.53*10^{4}}{m}$

Thus, $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\frac{1.53*10^4}{m}} = 4.1*10^{-4}\ m = \boxed{0.41\ mm}$

$v = \frac{\omega}{k} = \frac{\frac{2\pi*10^6}{s}}{\frac{1.53*10^4}{m}} \boxed{= 410\ \frac{m}{s}}$

The equivalent values in vacuum would be:

$\lambda = \frac{v}{f} = \frac{3.00*10^{8}\frac{m}{s}}{\frac{10^6}{s}} \boxed{= 300\ m}$ and $\boxed{v = 3.00*10^{8}\ \frac{m}{s}}$.

Then again, the wave is attenuating so quickly it hardly makes sense to call it a "wave".

Griffiths, Problem 9.21

(a) By combining Eq. 9.71 and Prob 9.12, we get:

$$\langle u \rangle = \frac{1}{4}\left(\varepsilon \tilde{\vec{E}}\cdot\tilde{\vec{E}}^* + \frac{1}{\mu}\tilde{\vec{B}}\cdot\tilde{\vec{B}}^*\right)$$

$$= \frac{1}{4}\left(\varepsilon E_0^2 e^{-2\kappa z} + \frac{1}{\mu} B_0^2 e^{-2\kappa z}\right)$$

$$= \frac{1}{4}\varepsilon E_0^2 e^{-2\kappa z}\left(1 + \frac{1}{\mu\varepsilon}\frac{B_0^2}{E_0^2}\right) \qquad \text{by Eq. 9.137}$$

$$= \frac{1}{4}\varepsilon E_0^2 e^{-2\kappa z}\left(1 + \frac{\varepsilon\mu}{\varepsilon\mu}\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right)$$

In a good conductor, $\frac{\varepsilon}{\sigma} << \frac{1}{\omega} \Rightarrow \frac{\sigma}{\varepsilon\omega} >> 1$ and the $\boxed{\text{second (magnetic) term completely dominates}}$.

From Eq. 9.126, $1 + \sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} = \frac{2k^2}{\omega^2\varepsilon\mu} \Rightarrow$

$$\langle u \rangle = \frac{1}{\cancel{4}_2}\cancel{\varepsilon} E_0^2 e^{-2\kappa z}\frac{\cancel{2}k^2}{\omega^2\cancel{\varepsilon}\mu} \boxed{= \frac{k^2}{2\mu\omega^2}E_0^2 e^{-2\kappa z}}, \text{ as given.}$$

(b) $$\vec{I} = \langle \vec{S} \rangle = \frac{1}{2\mu}\left(\tilde{\vec{E}}\times\tilde{\vec{B}}^*\right) = \frac{1}{2\mu}E_0 B_0 \cos\varphi\, e^{-2\kappa z}\,\hat{z}$$

$$= \frac{1}{2\mu}E_0^2\sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}\; e^{-2\kappa z}\cos\varphi\,\hat{z}$$

$$\cos\varphi = \cos\left(\tan^{-1}\left(\frac{\kappa}{k}\right)\right) = \frac{k}{\sqrt{k^2+\kappa^2}} = \frac{\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]^{1/2}}{\sqrt{2}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]^{1/2}}$$

$$\Rightarrow |I| = \frac{1}{2\mu}E_0^2\sqrt{\varepsilon\mu}\,\frac{\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]^{1/2}}{\sqrt{2}}\, e^{-2\kappa z} = \frac{1}{2\mu}E_0^2\frac{k}{\omega}e^{-2\kappa z}$$

$$\boxed{= \frac{k}{2\mu\omega}E_0^2 e^{-2\kappa z}}, \text{ as desired.}$$

Note that $\langle u \rangle v = I$, as expected.

Griffiths, Problem 9.25

For the present case, Eqs. 9.170 + 9.171 simplify to:

$$n \approx 1 + k\frac{(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \qquad \text{with "}k\text{"} = \frac{N q^2 f}{2m\varepsilon_0}$$

$$\alpha \approx \frac{2k}{c}\frac{\gamma\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

The extrema of n occur when $\frac{\partial n}{\partial(\omega^2)} = 0 \Rightarrow$

$$\left[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right](-1) - (\omega_0^2-\omega^2)\left[2(\omega_0^2-\omega^2)(-1)+\gamma^2\right] = 0$$

$$\Rightarrow -(\omega_0^2-\omega^2)^2 - \gamma^2\omega^2 + 2(\omega_0^2-\omega^2)^2 - (\omega_0^2-\omega^2)\gamma^2 = 0$$

$\Rightarrow (\omega_0^2-\omega^2)^2 - \omega_0^2\gamma^2 = 0 \Rightarrow \omega_0^2-\omega^2 = \pm\omega_0\gamma$, so the extrema occur at $\omega = \sqrt{\omega_0^2 \pm \omega_0\gamma} = \omega_0\sqrt{1 \pm \frac{\gamma}{\omega_0}} \approx \omega_0\left(1 \pm \frac{\gamma}{2\omega_0}\right)$.

The width of the anomalous dispersion region is $\boxed{\approx \gamma}$.

The absorption coefficient is a maximum for $\omega \approx \omega_0$ (when $\gamma \ll \omega_0$)

$$\Rightarrow \alpha_{MAX} \approx \frac{2k}{c}\frac{\gamma\omega_0^2}{\gamma^2\omega_0^2} = \boxed{\frac{2k}{c\gamma}}$$

The extrema of n occur where:

$$\alpha \approx \frac{2k}{c}\frac{\gamma(\omega_0^2 \pm \omega_0\gamma)}{\omega_0^2\gamma^2+\gamma^2(\omega_0^2\pm\omega_0\gamma)} = \frac{2k}{c}\frac{\gamma\omega_0^2\left(1\pm\frac{\gamma}{\omega_0}\right)}{\gamma^{\cancel{2}}\omega_0^2\left(\cancel{1}2\pm\frac{\gamma}{\omega_0}\right)}$$

$$= \frac{k}{c\gamma}\frac{1\pm\frac{\gamma}{\omega_0}}{1\pm\frac{\gamma}{2\omega_0}} = \frac{k}{c\gamma}\left(1\pm\frac{\gamma}{2\omega_0}\right) \approx \frac{1}{2}\left(\frac{2k}{c\gamma}\right) = \frac{\alpha_{MAX}}{2}.$$

Thus, the extrema occur where $\boxed{\alpha = \frac{1}{2}\alpha_{MAX}}$, as desired.

9.26.1

Griffiths, Problem 9.26

Define $\frac{Nq^2}{2m\varepsilon_0} = 0.003\omega_0^2 = \alpha\omega_0^2$. Then from Eq. 9.170:

$k = \frac{\omega}{c}\left[1 + \alpha\omega_0^2 \frac{(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\right]$. The group velocity is

$v_g = \frac{d\omega}{dk} \Rightarrow \frac{1}{v_g} = \frac{dk}{d\omega} \Rightarrow \frac{1}{y} = \frac{c}{v_g} = c\frac{dk}{d\omega}$. Thus,

$$\frac{1}{y} = 1 + \alpha\omega_0^2 \frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

$$+ \omega\alpha\omega_0^2 \frac{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2](-2\omega) - (\omega_0^2-\omega^2)[2(\omega_0^2-\omega^2)(-2\omega)+2\gamma^2\omega]}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^2}$$

$$= 1 + \alpha\omega_0^2 \frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

$$+ \alpha\omega_0^2 \frac{(-2)\gamma^2\omega^4 + 2(\omega_0^2-\omega^2)^2\omega^2 - (\omega_0^2-\cancel{\omega^2})2\gamma^2\omega^2}{[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^2}$$

$$= 1 + \alpha\omega_0^2 \frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

$$+ 2\alpha\omega_0^2 \frac{(\omega_0^2-\omega^2)^2\omega^2 - \gamma^2\omega^2\omega_0^2}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)^2}$$

Now let $x = \left(\frac{\omega}{\omega_0}\right)^2$ and redefine γ to take out $\omega_0 \Rightarrow$

$$\boxed{\frac{1}{y} = 1 + \alpha \frac{1-x}{(1-x)^2+\gamma^2 x} + 2\alpha \frac{(1-x)^2 x - \gamma^2 x}{[(1-x)^2+\gamma^2 x]^2}}$$

Griffiths asks for computer calculations, so I'm not going to bother simplifying further.

9.26.2

Additional calculations needed for the extra plots in part 6:

Eq. 9.170 gives $n = \frac{ck}{\omega} = \frac{c}{v_{ph}} = 1 + a\omega_0^2 \frac{(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega_0^2\omega^2}$

$$= 1 + a\frac{1-x}{(1-x)^2+\gamma^2 x}$$

. $\frac{v_{ph}}{c}$ is the inverse of this.

Also, Eq 9.171 gives:

$$\alpha = 2a\omega_0^2 \frac{\omega^2}{c} \frac{\gamma\omega_0}{(\omega_0^2-\omega^2)^2+\gamma^2\omega_0^2\omega^2}$$

$$= 2a\frac{\gamma x}{(1-x)^2+\gamma^2 x}\left(\frac{\omega_0}{c}\right)$$

. $\frac{\omega_0}{c}$ specifies the natural unit for α. Note that this $= k$ for a wave with ω_0 in vacuum.

$a := 0.003$

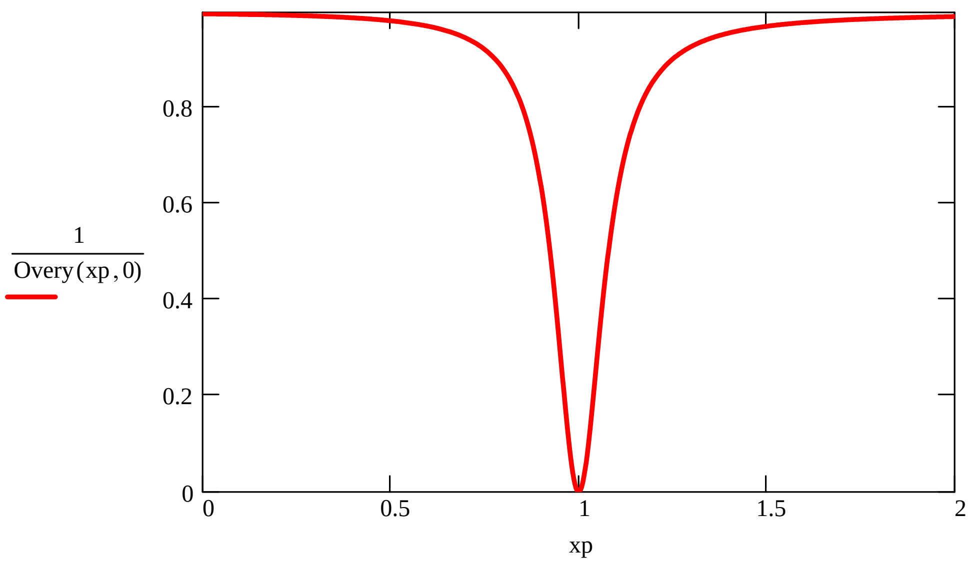
Overy is 1/y, so below I will plot 1/Overy vs. x

$$\mathrm{Overy}(x,\mathrm{gamma}) := 1 + a\cdot\frac{1-x}{(1-x)^2+\mathrm{gamma}^2\cdot x} + 2\cdot a\cdot\frac{(1-x)^2\cdot x-\mathrm{gamma}^2\cdot x}{\left[(1-x)^2+\mathrm{gamma}^2\cdot x\right]^2}$$

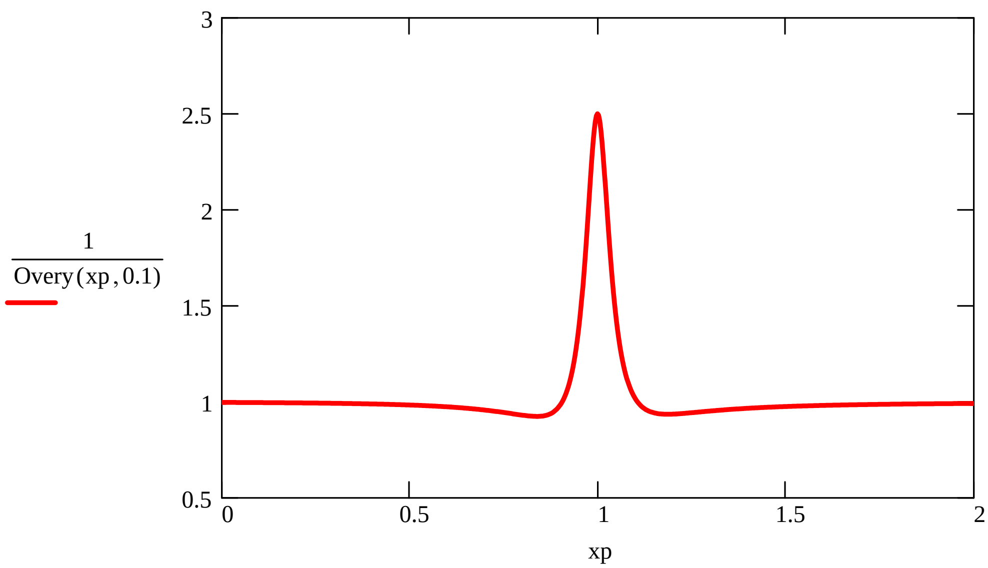
For gamma=0, Overy diverges at x=1. Therefore, I'm choosing my points to plot in such a way that I avoid that particular value.

$xp := 0, 0.003 .. 2$

Here is the part (a), gamma=0, plot:



Here is the part (b), gamma=0.1, plot:

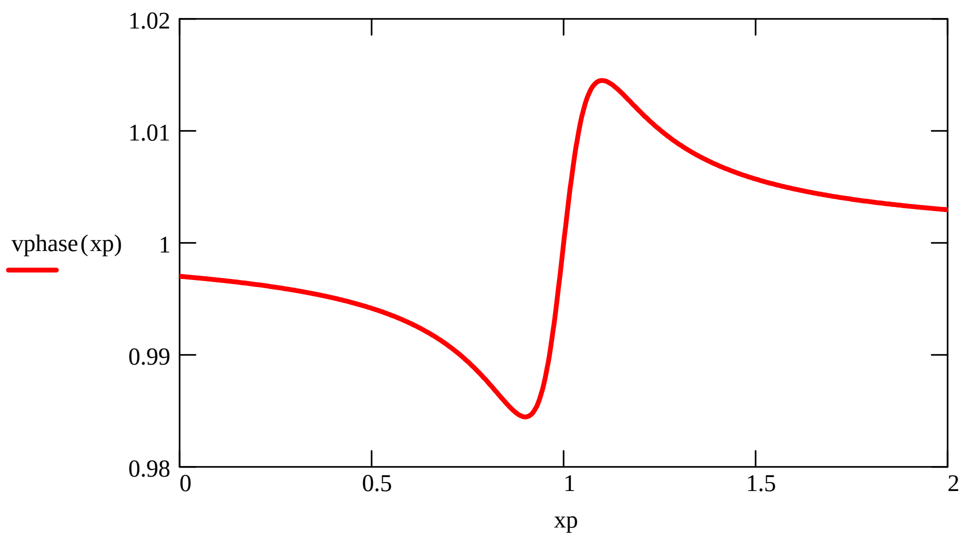


gamma := 0.1

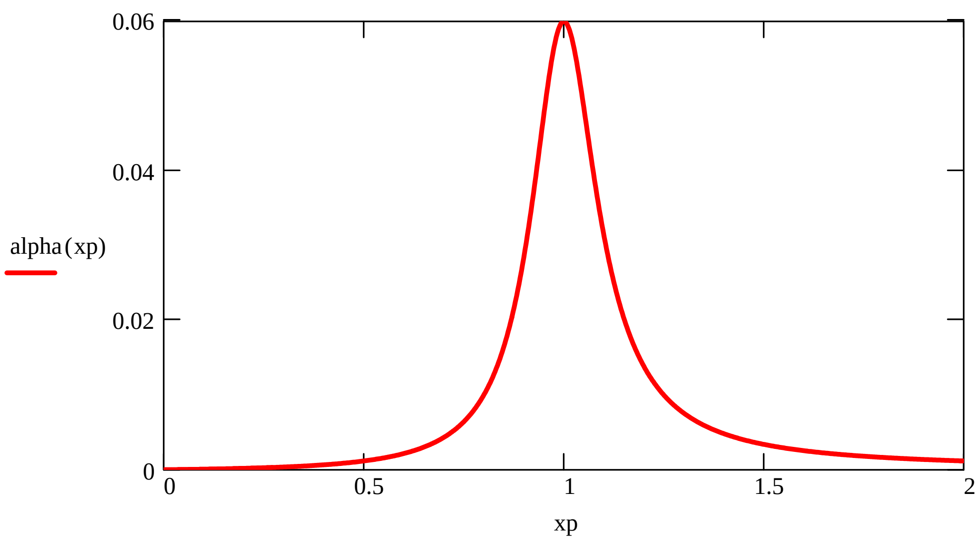
$$\mathrm{vphase}(x) := \frac{1}{1 + a\cdot\frac{1-x}{(1-x)^2 + \mathrm{gamma}^2\cdot x}}$$

$$\mathrm{alpha}(x) := 2\cdot a\cdot\frac{\mathrm{gamma}\cdot x}{(1-x)^2 + \mathrm{gamma}^2\cdot x}$$

Here is the phase velocity for part (b), in units c:



Here is the attenuation length for part (b), in units of omega_0/c:



9.39.1

Griffiths, Problem 9.39

(a) $\tilde{\vec{E}}_T(\vec{r},t) = \tilde{E}_{0_T} e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$

with $\vec{k}_T = k_T(\sin\theta_T\hat{x}+\cos\theta_T\hat{z}) = \frac{\omega n_2}{c}\left[\frac{n_1}{n_2}\sin\theta_I\hat{x} + i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_I - 1}\,\hat{z}\right]$

$= \frac{\omega n_1}{c}\sin\theta_I\hat{x} + i\frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_I - n_2^2}\,\hat{z}$

$\Rightarrow e^{i(k_T\cdot\vec{r}-\omega t)} = e^{-\frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_I-n_2^2}\,z}\, e^{i\left(\frac{\omega}{c}n_1\sin\theta_I x - \omega t\right)}$

$\Rightarrow \boxed{K = \frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_I - n_2^2} \text{ and } k = \frac{\omega}{c}n_1\sin\theta_I}$, as desired.

(b) Eq. 9.109 gives $R = \frac{|E_{0_R}|^2}{|E_{0_I}|^2} = \left|\frac{\alpha-\beta}{\alpha+\beta}\right|^2$

But $\alpha = i\gamma$, where β & γ are real, so $R = \left|\frac{i\gamma-\beta}{i\gamma+\beta}\right|^2 = \frac{\gamma^2+\beta^2}{\gamma^2+\beta^2} = 1$, as desired.

(c) Problem 9.17 gave $R = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right|^2$. Once again, $\alpha = i\gamma$ gives β & γ real. Then $R = \left|\frac{1-i\beta\gamma}{1+i\beta\gamma}\right|^2 = \frac{1+(\beta\gamma)^2}{1+(\beta\gamma)^2} = 1$, as desired.

(d) Take the phase of $\tilde{\vec{E}}_{0_T}$ in Eq. 9.201 to be real, and $|\tilde{\vec{E}}_{0_T}| = E_0$.

Then $\vec{E} = \mathrm{Re}(\text{Eq. } 9.201) = \boxed{E_0 e^{-Kz}\cos(kx-\omega t)\,\hat{y}},$ as desired.

$\tilde{\vec{B}} = \frac{\vec{k}_T}{\omega}\times\tilde{\vec{E}} = \frac{1}{\omega}(k\hat{x}+iK\hat{z})\times E_0\hat{y}e^{-Kz}e^{i(kx-\omega t)}$.

$\hat{x}\times\hat{y} = \hat{z}$ and $\hat{z}\times\hat{y} = -\hat{x} \Rightarrow$

$\tilde{\vec{B}} = \frac{E_0}{\omega}e^{-Kz}\left[k e^{i(kx-\omega t)}\hat{z} - iKe^{i(kx-\omega t)}\hat{x}\right]$. Taking the real part:

$\boxed{\vec{B} = \frac{E_0}{\omega}e^{-Kz}\left[k\cos(kx-\omega t)\hat{z} + K\sin(kx-\omega t)\hat{x}\right]}$, as desired.

9.39.2

(e) $\vec{E} = \vec{E}(x,z,t)\,\hat{y} \Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = 0}$ ✓

$$\vec{\nabla}\cdot\vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} = \frac{E_0}{\omega}\Big\{ Ke^{-Kz}\big(k\cos(kx-\omega t)\big)$$

$$+ \left(-Ke^{-Kz}\right)\big(k\cos(kx-\omega t)\big)\Big\} = 0 \Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$ ✓

$$\vec{\nabla}\times\vec{E} = \frac{\partial E_y}{\partial x}\hat{z} - \frac{\partial E_y}{\partial z}\hat{x}$$

FROM TWO MINUS SIGNS

$$= -E_0 k e^{-Kz}\sin(kx-\omega t)\,\hat{z} + E_0 K e^{-Kz}\cos(kx-\omega t)\,\hat{x}$$

$$-\frac{\partial\vec{B}}{\partial t} = -(-\omega)\frac{E_0}{\omega}e^{-Kz}\Big[K\cos(kx-\omega t)\,\hat{x} - k\sin(kx-\omega t)\,\hat{z}\Big]$$

$$= E_0 e^{-Kz}\Big[K\cos(kx-\omega t)\,\hat{x} - k\sin(kx-\omega t)\,\hat{z}\Big] = \vec{\nabla}\times\vec{E} \Rightarrow$$

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$ ✓

$$\vec{\nabla}\times\vec{B} = \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\hat{y}$$

$$= \frac{E_0}{\omega}\Big[\left(-Ke^{-Kz}\right)\big(K\sin(kx-\omega t)\big) - \left(ke^{-Kz}\right)\big(-k\sin(kx-\omega t)\big)\Big]\hat{y}$$

$$= \frac{E_0}{\omega}\left(k^2 - K^2\right)e^{-Kz}\sin(kx-\omega t)\,\hat{y}$$

But $k^2 - K^2 = \frac{\omega^2}{c^2}n_1^2\sin^2\theta_I - \frac{\omega^2}{c^2}\left(n_1^2\sin^2\theta_I - n_2^2\right) = \frac{\omega^2 n_2^2}{c^2} = \omega^2\mu\varepsilon$

$\Rightarrow \vec{\nabla}\times\vec{B} = \omega E_0 \mu\varepsilon\, e^{-Kz}\sin(kx-\omega t)\,\hat{y}$. Meanwhile,

$$\frac{\partial\vec{E}}{\partial t} = E_0 e^{-Kz}\Big(\underbrace{(-\omega)\big(-\sin(kx-\omega t)\big)}_{\text{CANCEL}}\Big)\hat{y} \Rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu\varepsilon\frac{\partial\vec{E}}{\partial t}}$$ ✓.

(f) $\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu}(\vec{E} \times \vec{B})$

$$\boxed{= \frac{E_0^2}{\mu\omega} e^{-2Kz}\left[-K\sin(kx-\omega t)\cos(kx-\omega t)\,\hat{z} + k\cos^2(kx-\omega t)\,\hat{x}\right]}$$

$\langle \cos^2 \rangle = \frac{1}{2}$ and $\langle \sin\cos \rangle = 0 \Rightarrow$

$$\boxed{\langle \vec{S} \rangle = \frac{E_0^2 k}{2\mu\omega} e^{-2Kz}\,\hat{x}}$$

has no component in the $\hat{z}$ direction. Energy flows only in the $(+)\hat{x}$ direction, as we expected/hoped.