Physics 305 – Homework Set 2

Due in class on Wednesday, Jan. 29

Do the following eight problems from Griffiths:

7.3 (pg 302).

7.4 (pg 303).

7.8 (pg 311).

7.10 (pg 311).

7.12 (pg 316).

For this problem, assume that ω is sufficiently small so that the magnetic field at time t differs only negligibly from the magnetostatic solution. Also do the following part (b) for this question:

(b) Repeat your calculation for a resistive loop of radius $2a$ encircling and coaxial with the solenoid.

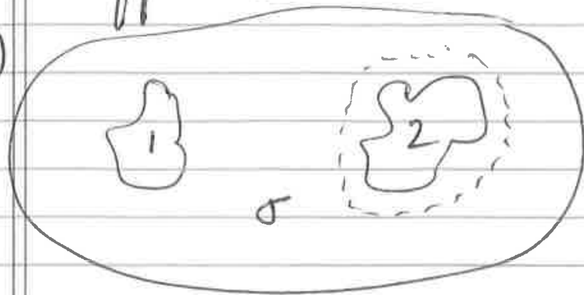
7.19 (pg 321).

7.22 (pg 327).

7.26 (pg 327). This is for you to see some real numbers.

Griffith, Problem 7,3

(a)



Consider a closed (dashed) surface surrounding object 2.

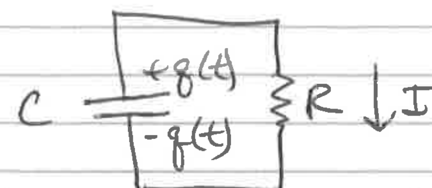
$$I = \oint_S \vec{J}\cdot d\vec{a} = \oint_S \sigma \vec{E}\cdot da$$

$$= \sigma \oint_S \vec{E}\cdot d\vec{a} = \frac{\sigma}{\varepsilon_0} Q_2$$, from Gauss' Law.

The capacitance gives $Q_2 = CV \Rightarrow I = \frac{\sigma C}{\varepsilon_0} V \Rightarrow$

$V = \frac{\varepsilon_0}{\sigma C} I = RI \Rightarrow \boxed{R = \frac{\varepsilon_0}{\sigma C}}$ as desired.

(b) We have an R-C circuit:



In this circuit $V(t) = \frac{q(t)}{C} = IR$ and $I = -\frac{dq}{dt} \Rightarrow$

$$R\frac{dq}{dt} + \frac{1}{C} q = 0 \Rightarrow \frac{dq}{dt} + \frac{1}{RC} q = 0$$

$$\Rightarrow q(t) = Q_0 e^{-\frac{t}{RC}} \Rightarrow \boxed{V = V_0 e^{-\frac{t}{RC}}} \text{ with } \boxed{\tau = RC = \frac{\varepsilon_0}{\sigma}}.$$

Griffiths, Problem 7.4

$I(s) = \oint \vec{J} \cdot d\vec{a} = J(s)\, 2\pi s L = \sigma(s) \hat{E}(s)\, 2\pi s L = \text{constant}$,

for a steady-state.

$\Rightarrow \sigma(s) E(s) = \frac{I}{2\pi s L}$. $\sigma(s) = \frac{k}{s} \Rightarrow E(s) = \frac{I}{2\pi k L}$, independent of s!

Thus, $V = -\int_b^a \vec{E} \cdot d\vec{l} = \frac{I}{2\pi k L}(b-a) = IR$

$$\Rightarrow R = \frac{b-a}{2\pi k L}$$

Side note: $E(s) = \text{constant}$ means $\vec{\nabla} \cdot \vec{E} \neq 0$. Therefore, this case requires a finite static charge density within the volume of the cylindrical conducting shell. This does not violate our standard rules for conductors because:

a) σ is finite here, and

b) an external source must be providing the new charge (and energy) necessary to maintain the voltage difference between the inner and outer cylinders.

Griffiths, Problem 7.8

(a) $\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi} \Rightarrow \Phi = \int \vec{B}\cdot d\vec{a} = \frac{\mu_0 I}{2\pi} a \int_s^{s+a} \frac{ds}{s} = \boxed{\frac{\mu_0 I a}{2\pi} \ln \frac{s+a}{s}}$

(b) $\frac{d\Phi}{dt} = \frac{d\Phi}{ds}\frac{ds}{dt} = \frac{\mu_0 I a}{2\pi}\left(\frac{1}{s+a} - \frac{1}{s}\right) v = \frac{\mu_0 I a}{2\pi}\left[\frac{s-s-a}{s(s+a)}\right] v$

$= -\frac{\mu_0 I a^2 v}{2\pi} \frac{1}{s(s+a)}$

$\Rightarrow \boxed{\mathcal{E} = \frac{\mu_0 I a^2 v}{2\pi}\frac{1}{s(s+a)}}$ The flux is decreasing. So the emf will try to add to the existing $\vec{B}$ field. The initial field is out of the page, so the induced current will be **counterclockwise**.

(c) In this case, $\frac{d\Phi}{dt} = 0 \Rightarrow \boxed{\mathcal{E} = 0}$ and there is **no induced current**.

Griffiths, Problem 7.10

$$\int \vec{B}\cdot d\vec{a} = Ba^2\,\hat{B}\cdot\hat{a} = Ba^2 \sin(\omega t) = \Phi$$

$$\Rightarrow \ \mathcal{E} = -\frac{d\Phi}{dt} = -Ba^2\omega\cos(\omega t)$$

Griffiths, Problem 7.12 (plus extra part)

(a) $\Phi(t) = \int \vec{B}\cdot d\vec{a} = B_0 \cos(\omega t)\,\pi\left(\frac{a}{2}\right)^2 = \frac{\pi B_0 a^2}{4}\cos\omega t$

$\Rightarrow \mathcal{E} = -\frac{d\Phi}{dt} = \frac{\pi B_0 a^2 \omega}{4}\sin(\omega t)$

$\Rightarrow I(t) = \frac{\mathcal{E}(t)}{R} = \boxed{\frac{\pi B_0 a^2 \omega}{4R}\sin(\omega t)}$

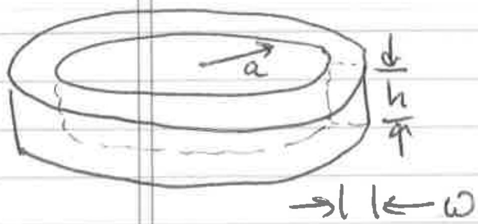
(b) There is no magnetic field outside a long solenoid. Thus, in this case:

$\Phi(t) = B_0 \cos(\omega t)\,\pi a^2$

This is 4 times the previous case, so

$$\boxed{I(t) = \frac{\pi B_0 a^2 \omega}{R}\sin(\omega t)}$$

Griffiths, Problem 7.19



$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow B\,2\pi s = \mu_0 N I$

$\Rightarrow B(s) = \frac{\mu_0 N I}{2\pi s}$ inside the toroid, and 0 outside

$\Rightarrow \frac{\partial B}{\partial t} = \frac{\mu_0 N}{2\pi s}k$ inside the toroid.

Following the discussion on pg 317, we can write:

around the loop

$$\vec{E} = \frac{1}{4\pi}\int \frac{-\frac{\partial \vec{B}}{\partial t}\times\hat{r}}{r^2}\,d\tau \,.\quad d\tau = s\,ds\,d\phi\,dz \approx ds\,dz\,dl'$$

a is much greater than h and w, so we can collapse the toroid down to a ring for purpose of calculating $\vec{E}$ along the axis. Then:

$$\vec{E} \approx \frac{-1}{4\pi}\int\frac{\left(\int\frac{\partial\vec{B}}{\partial t}ds\,dz\right)\times\hat{r}}{r^2}\,dl' = \frac{-1}{4\pi}\frac{\mu_0 N k h}{2\pi}\ln\left(\frac{a+w}{a}\right)\int\frac{d\vec{l}'\times\hat{r}}{r^2}$$

$$\boxed{\Rightarrow \vec{E}(z) \approx \frac{-\mu_0 N k h}{4\pi}\ln\left(\frac{a+w}{a}\right)\frac{a^2}{(a^2+z^2)^{3/2}}\hat{z}}$$ from Eq 5.41

Griffiths, Problem 7.22

a) From Eq. 5.41, $B(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+z^2)^{3/2}}$ on axis. We can treat this as constant over the small loop, so:

$$\boxed{\Phi = \frac{\mu_0 \pi a^2}{2} \frac{b^2}{(b^2+z^2)^{3/2}} I_{big}}$$

b) $\Phi = \oint \vec{A} \cdot d\vec{l}$, where from Eq. 5.87, $\vec{A}_{dip} = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\phi}$

We have $r^2 = b^2 + z^2$ and $\sin\theta = \frac{b}{\sqrt{b^2+z^2}} \Rightarrow$

$$\Phi = \frac{\mu_0}{4\pi} (I_{small} \pi a^2) \frac{b}{(b^2+z^2)^{3/2}} \oint \hat{\phi} \cdot d\vec{l}$$

$$= \frac{\mu_0}{\cancel{4\pi}_{\,2}} I_{small} \pi a^2 \frac{b}{(b^2+z^2)^{3/2}} \cancel{2\pi} b = \boxed{\frac{\mu_0 \pi a^2}{2} \frac{b^2}{(b^2+z^2)^{3/2}} I_{small}}$$

c) $\Phi_2 = M_{21} I_1$ gives

$$\boxed{M = \frac{\mu_0 \pi a^2}{2} \frac{b^2}{(b^2+z^2)^{3/2}}}$$ in both cases.

Griffiths, Problem 7.26

a) $\vec{B} = \frac{\mu_0 I_0 \cos(\omega t)}{2\pi s}\hat{\varphi} \Rightarrow$ for one turn of the toroid:

$$\Phi = \int_{s_{MIN}}^{s_{MAX}} \frac{\mu_0 I_0 \cos(\omega t)}{2\pi s} h\,ds = \frac{\mu_0 I_0 h}{2\pi} \ln\left(\frac{s_{MAX}}{s_{MIN}}\right)\cos(\omega t)$$

$$\Rightarrow \mathcal{E} = -N\frac{d\Phi}{dt} = \frac{\mu_0 N I_0 h}{2\pi}\ln\left(\frac{s_{MAX}}{s_{MIN}}\right)\omega \sin(\omega t)$$

$$= \frac{4\cancel{\pi} * 10^{-7}\frac{N}{A^2}}{2\cancel{\pi}}(1000)(0.5A)(0.01m)\ln(2)\frac{2\pi * 60}{s}\sin(\omega t)$$

$$\boxed{= (2.61 * 10^{-4}\,V)\sin(\omega t)}$$

$$I_R = \frac{\mathcal{E}}{R} = \frac{2.61 * 10^{-4}\,V}{500\,\Omega}\sin(\omega t) = \boxed{(5.23 * 10^{-7}A)\sin(\omega t)}$$

b) From Eq. 5.60, the field in the toroid produced by I_R is:

$\vec{B} = \frac{\mu_0 N I_R(t)}{2\pi s}\hat{\varphi} \Rightarrow$ for one turn of the toroid:

$$\Phi = \frac{\mu_0 N I_R h}{2\pi}\ln\left(\frac{s_{MAX}}{s_{MIN}}\right)\sin(\omega t) \Rightarrow$$

$$\mathcal{E}_{back} = -N\frac{d\Phi}{dt} = -\frac{\mu_0 N^2 I_R h}{2\pi}\ln\left(\frac{s_{MAX}}{s_{MIN}}\right)\omega\cos(\omega t) \Rightarrow$$

The ratio of amplitudes is: $\left|\frac{\mathcal{E}_{back}}{\mathcal{E}}\right| = \frac{N I_R}{I_0} = \frac{1000 * 5.23 * 10^{-7}A}{0.5\,A}$

$$= \boxed{1.05 * 10^{-3}}$$

Note: If $\left|\frac{\mathcal{E}_{back}}{\mathcal{E}}\right|$ had been significant, it would have been important to consider $\mathcal{E}_{back}$ in the original calculation of I_R.