# Physics 305 – Homework Set 1
**Due in class on Wednesday, Jan. 22**

Do the following six problems from Griffiths:

5.23 (pg 248)

5.27 (pg 248)

6.10 (pg 277)

6.12 (pg 282)

6.13 (pp 282-83)

6.16 (pg 287)

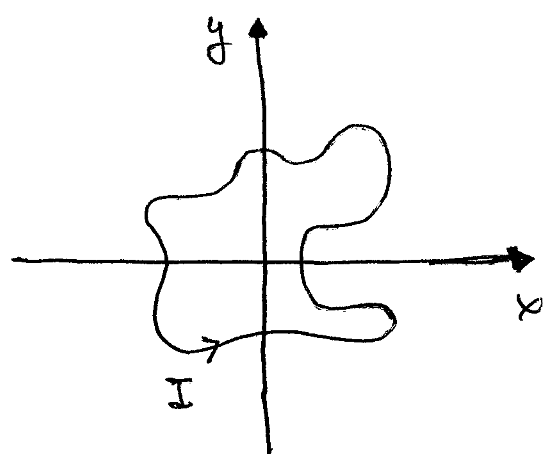
Also do the following additional problem (that includes aspects of several Griffiths problems):

The figure below shows a coil of wire located in the $x$-$y$ plane and encircling the $z$-axis. The coil is carrying a current $I$ in the $x$-$y$ plane, as shown. The coil has area $a$, and is entirely contained within a circle of radius $R$. Note that the shape of the coil does NOT obey any particular symmetry.

a) Assume that there are many such coils placed one on top of the other, with $n$ coils per unit length, to form a finite solenoid-like magnet along the $z$-axis that extends from $z = -L$ to $z = +L$. Use general properties of the vector potential for this problem to show that the magnetic field points in the $z$ direction at all points in the $x$-$y$ plane.

b) Approximate the magnetic field in the $s >> R$ region of the $x$-$y$ plane. **Note**: Do NOT make any assumption about the relative magnitude of $s$ relative to $L$.

c) Now consider the limiting case where $L \to \infty$. Find the exact magnetic field at all points in space.



Griffiths, Problem 5.23

$$\vec{A} = \frac{\mu_0 I}{4\pi}\hat{z}\int_{z_1}^{z_2}\frac{dz'}{\sqrt{s^2+(z-z')^2}} = \frac{\mu_0 I}{4\pi}\hat{z}\int_{z_1}^{z_2}\frac{dz'}{\sqrt{s^2+(z'-z)^2}}$$

$$= \frac{\mu_0 I}{4\pi}\hat{z}\int_{z_1-z}^{z_2-z}\frac{dy}{\sqrt{s^2+y^2}} = \frac{\mu_0 I}{4\pi}\hat{z}\left[\ln\left[y+\sqrt{y^2+s^2}\right]\right]_{y=z_1-z}^{z_2-z}$$

$$\Rightarrow \boxed{\vec{A} = \frac{\mu_0 I}{4\pi}\ln\frac{(z_2-z)+\sqrt{(z_2-z)^2+s^2}}{(z_1-z)+\sqrt{(z_1-z)^2+s^2}}\,\hat{z}} = A_z(s,z)\hat{z}$$

$\vec{B} = \vec{\nabla}\times\vec{A}$. For $\vec{A} = A_z(s,z)\hat{z}$, this reduces to:

$$B = -\frac{\partial A_z}{\partial s}\hat{\varphi}$$

$$= \frac{-\mu_0 I}{4\pi}\left\{\frac{1}{(z_2-z)+\sqrt{(z_2-z)^2+s^2}}\,\frac{1}{\cancel{2}}\,\frac{\cancel{2}s}{\sqrt{(z_2-z)^2+s^2}} - \frac{1}{(z_1-z)+\sqrt{(z_1-z)^2+s^2}}\,\frac{1}{\cancel{2}}\,\frac{\cancel{2}s}{\sqrt{(z_1-z)^2+s^2}}\right\}\hat{\varphi}$$

$$\boxed{= -\frac{\mu_0 I s}{4\pi}\left\{\frac{1}{\left[(z_2-z)+\sqrt{(z_2-z)^2+s^2}\right]\sqrt{(z_2-z)^2+s^2}} - \frac{1}{\left[(z_1-z)+\sqrt{(z_1-z)^2+s^2}\right]\sqrt{(z_1-z)^2+s^2}}\right\}\hat{\varphi}}$$

5.23.2

This doesn't look anything like Eq. 5.37[redacted], which can be written as:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \left\{ \frac{z_2 - z}{\sqrt{(z_2 - z)^2 + s^2}} - \frac{z_1 - z}{\sqrt{(z_1 - z)^2 + s^2}} \right\} \hat{\varphi} .$$

After playing with the two expressions for a while – and checking my arithmetic three (or more!) times – I punted. I plugged these two expressions into MathCad, and calculated their difference. It came up zero to within numerical round-off error. So we do get the same $\vec{B}$ field either way, as we must, even though the two answers appear very different.

Griffiths, Problem 5.27

$\vec{J}$ goes to infinity, so I can't set $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{\imath} d\tau$.

$\vec{J} = K\hat{x}$, so let's assume $\vec{A} = A_x \hat{x}$. The problem is invariant under translations in $x+y$, so $\vec{A} = A_x(z)\hat{x}$. Then

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\partial A_x}{\partial z}\hat{y} = \begin{cases} -\frac{\mu_0}{2}K\hat{y} & \text{for } z>0 \\ +\frac{\mu_0}{2}K\hat{y} & \text{for } z<0 \end{cases}.$$

If we set $A_x(z=0) = 0$, then:

$$z>0: \quad A_x(z) = \int_0^z -\frac{\mu_0}{2}K\,dz = -\frac{\mu_0}{2}Kz = -\frac{\mu_0}{2}K|z|$$

$$z<0: \quad A_x(z) = \int_0^z \frac{\mu_0 K}{2}\,dz = \frac{\mu_0}{2}Kz = -\frac{\mu_0}{2}K|z| \Rightarrow$$

$$\vec{A} = -\frac{\mu_0}{2}K|z|\hat{x}$$

Griffiths, Problem 6.10

First, let's calculate $\vec{B}$ if there was no gap. Then $\vec{K}_b = \vec{M} \times \hat{n}$ provides a "toroidal" current. The magnetic field in a toroid is $\frac{\mu_0 N I}{2\pi s}\hat{\phi}$ from Eq. 5.60. $NI$ is the total current flowing around the toroid. In this case, that becomes $K_b 2\pi s = M 2\pi s \Rightarrow \vec{B} = \mu_0 M \hat{\phi}$. (We get the same answer by setting $\vec{H} = 0 = \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \vec{B} = \mu_0 \vec{M}$.)

From the hint, we must add the magnetic field from four line segments with current $\vec{I} = -\omega \vec{K}_b$. In the units of Eq. 5.37, each line segment runs from $\theta_1 = -45° \rightarrow \theta_2 = +45° \Rightarrow$

$$\vec{B}_{square} = -\frac{\mu_0 \omega M}{4\pi \frac{a}{2}}\left(\frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}}\right) 4\hat{\phi} = \frac{-\mu_0 \omega M}{\pi a}\frac{4}{\sqrt{2}}\hat{\phi}\frac{\sqrt{2}}{\sqrt{2}}$$

$$= -\frac{\mu_0 \omega M 2\sqrt{2}}{\pi a}\hat{\phi} \Rightarrow$$

$$\vec{B}_{center\ of\ gap} = \mu_0 M \hat{\phi}\left[1 - \frac{2\sqrt{2}}{\pi}\frac{\omega}{a}\right]$$

Griffiths, Problem 6.12

(a) $\vec{J}_b = \vec{\nabla} \times \vec{M} = \vec{\nabla} \times (ks\hat{z}) = -k\hat{\varphi}$

$\vec{K}_b = \vec{M} \times \hat{n} = ks\hat{z} \times \hat{s} = ks\hat{\varphi}$

$\vec{K}_b$ represents an infinite solenoid with $nI = kR \Rightarrow \vec{B} = \mu_0 kR\hat{z}$ inside (0 outside)

$\vec{J}_b$ constitutes an infinite set of infinite solenoids, such that the net field at $s$ $(<R)$ is $\int_s^R -\mu_0 k\,ds = -\mu_0 ks\Big]_s^R = (\mu_0 ks - \mu_0 kR)\hat{z}$

Adding these together to get the total field, we find:

$$\boxed{\vec{B} = \begin{cases} \mu_0 ks\hat{z} & \text{inside } (s<R) \\ 0 & \text{outside } (s>R) \end{cases}}$$

(b) There is no free current, so $\vec{H} = 0 \Rightarrow \boxed{\vec{B} = \mu_0\vec{M} = \begin{cases} \mu_0 ks\hat{z} & (s<R) \\ 0 & (s>R) \end{cases}}$

We get the same answer, a lot faster.

Griffiths, Problem 6.13

(a) From Eq. 6.16, $\vec{B}$ from the magnetized sphere is $\vec{B} = \frac{2}{3}\mu_0 \vec{M}$.

Thus, $\boxed{\vec{B}_{NEW} = \vec{B}_0 - \frac{2}{3}\mu_0 \vec{M}.}$

$$\vec{H}_{NEW} = \frac{1}{\mu_0}\vec{B}_{NEW} = \frac{1}{\mu_0}\vec{B}_0 - \frac{2}{3}\vec{M} = \left(\vec{H}_0 + \vec{M}\right) - \frac{2}{3}\vec{M} = \boxed{\vec{H}_0 + \frac{1}{3}\vec{M}}$$

(b) $\vec{B}$ in an (infinite) magnetized rod is $\vec{B}_{ROD} = \mu_0 \vec{M}$. Thus,

$\boxed{\vec{B}_{NEW} = \vec{B}_0 - \mu_0 \vec{M}}$.

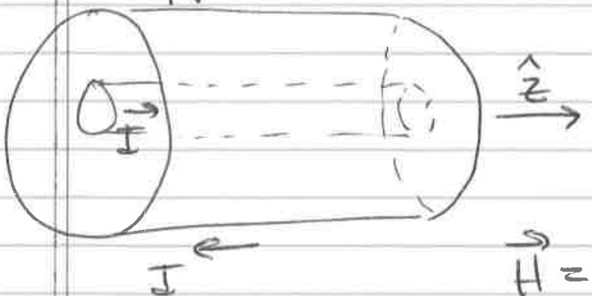
$$\vec{H}_{NEW} = \frac{1}{\mu_0}\vec{B}_{NEW} = \frac{1}{\mu_0}\vec{B}_0 - \vec{M} = \left(\vec{H}_0 + \vec{M}\right) - \vec{M} = \boxed{\vec{H}_0}.$$

(c) $\vec{B}$ at the center of a thin magnetized wafer is just that of a current loops with $I = M *$ thickness at the center. For a sufficiently large and thin wafer, this $\rightarrow 0 \Rightarrow$

$\boxed{\vec{B}_{NEW} = \vec{B}_0}$.

$$\vec{H}_{NEW} = \frac{1}{\mu_0}\vec{B}_{NEW} = \frac{1}{\mu_0}\vec{B}_0 = \boxed{\vec{H}_0 + \vec{M}}.$$

Griffiths, Problem 6.16




$\oint \vec{H} \cdot d\vec{l} = I_{f,encl} \Rightarrow$

for $a < s < b$, $H(2\pi s) = I \Rightarrow$

$\vec{H} = \frac{I}{2\pi s}\hat{\varphi} \Rightarrow \vec{B} = (1+\chi_m)\mu_0\vec{H} = (1+\chi_m)\frac{\mu_0 I}{2\pi s}\hat{\varphi}$

($\vec{B} = 0$ and $\vec{H} = 0$ for $s < a$ and $s > b$.)

$\vec{M} = \chi_m \vec{H} = \frac{\chi_m I}{2\pi s}\hat{\varphi} \Rightarrow \vec{J}_b = \vec{\nabla} \times \vec{M} = 0$

$\vec{K}_b = \vec{M} \times \hat{n}$

At $s = a$, $\hat{n} = -\hat{s} \Rightarrow \vec{K}_b = \frac{\chi_m I}{2\pi a}(\hat{\varphi} \times (-\hat{s})) = \frac{\chi_m I}{2\pi a}\hat{z}$

At $s = b$, $\hat{n} = \hat{s} \Rightarrow \vec{K}_b = \frac{\chi_m I}{2\pi b}\hat{\varphi} \times \hat{s} = -\frac{\chi_m I}{2\pi b}\hat{z}$

Thus, at $s = a$, $I_{tot} = I_f + I_b = I + 2\pi a\left(\frac{\chi_m I}{2\pi a}\right) = (1+\chi_m)I$

At $s = b$, $I_{tot} = -I + 2\pi b\left(-\frac{\chi_m I}{2\pi b}\right) = (1+\chi_m)(-I)$.

The $\vec{B}$ field is the same as that given by $\pm$ currents $\pm(1+\chi_m)I$ in vacuum, so they match. ✓

# Homework Set 10, Extra Problem

(a) $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{r} d\tau' = \frac{\mu_0}{4\pi}\int \frac{\vec{I}n}{r} dl' dz'$ for this configuration.

$\vec{I}\cdot\hat{z} = 0 \Rightarrow \vec{A} = A_\varphi \hat{\varphi} + A_s \hat{s}$ <u>everywhere</u>. Meanwhile:

$$\vec{A}(x,y,-z) = \frac{\mu_0}{4\pi}\oint\int_{-L}^{L} \frac{\vec{I}n \quad dl' dz'}{\sqrt{(x-x')^2+(y-y')^2+(-z-z')^2}} \qquad \text{Let } z'' = -z' \Rightarrow$$

$$= \frac{\mu_0}{4\pi}\oint\int_{+L}^{-L} \frac{\vec{I}n \quad dl'(-dz'')}{\sqrt{(x-x')^2+(y-y')^2+(-z+z'')^2}}$$

$$= \frac{\mu_0}{4\pi}\oint\int_{-L}^{L} \frac{\vec{I}n \quad dl' dz''}{\sqrt{(x-x')^2+(y-y')^2+(z-z'')^2}} = \vec{A}(x,y,z).$$

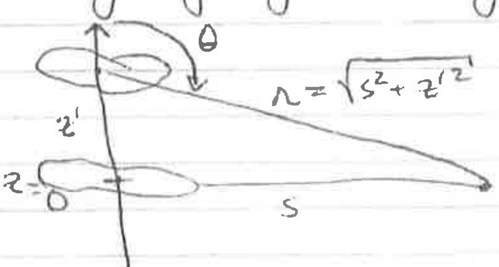
Thus, at $z=0$, $\frac{\partial \vec{A}}{\partial z} = 0 \Rightarrow \frac{\partial A_\varphi}{\partial z} = 0$ and $\frac{\partial A_s}{\partial z} = 0$. (In general, the second derivatives, and $4^{th}$, $6^{th}$, $8^{th}$, ..., will be non-zero.).

Combining those with $A_z = 0$ everywhere gives:

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\varphi) - \frac{\partial A_s}{\partial \varphi}\right)\hat{z} \Rightarrow \boxed{\vec{B} \parallel \hat{z}}.$$

(b) For $s \gg R$, we can approximate $\vec{A}$ for any given loop with just the magnetic dipole term. Adding the many loops together, we get

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{-L}^{L} \frac{Ian\,\hat{z}\times\hat{r}}{r^2} dz'$$




$$\hat{z}\times\hat{r} = \sin\theta\,\hat{\phi} = \frac{s}{r}\hat{\phi} \Rightarrow$$

$$\vec{A} = \frac{\mu_0 Ians}{4\pi}\hat{\phi}\int_{-L}^{L} \frac{dz'}{(s^2+z'^2)^{3/2}} = \frac{\mu_0 Ian\cancel{s}}{4\pi}\left[\frac{z'}{s^{\cancel{2}}\sqrt{s^2+z'^2}}\right]_{z'=-L}^{L}\hat{\phi}$$

Integral #141

HW10.EP.2
[redacted]

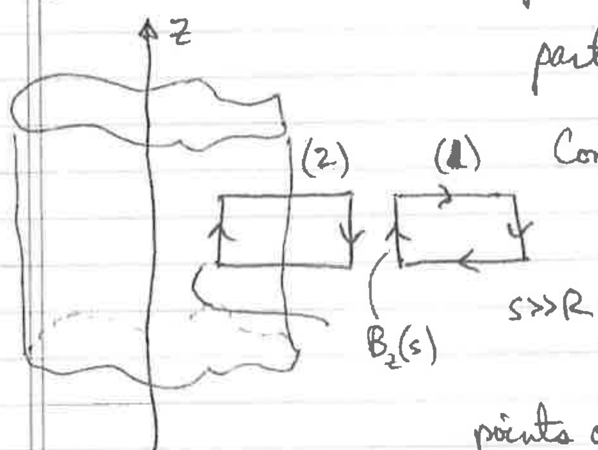
$$= \frac{\mu_0 I a n}{2\cancel{4}\pi s}\,\frac{\cancel{2}L}{\sqrt{s^2+L^2}}\,\hat{\phi}.$$

This result is only valid for $z=0$. But from part a, $\vec{B} = B_z\hat{z}$, so we can ignore any $z$-dependence of $\vec{A}$ when calculating $\vec{B}$ for $z=0$.

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0 I a n L}{2\pi}\,\frac{1}{s}\left[\frac{\partial}{\partial s}\left(\cancel{s}\cdot\frac{1}{\cancel{s}\sqrt{s^2+L^2}}\right)\right]\hat{z}$$

$$= \frac{\mu_0 I a n L}{2\pi \cancel{s}}\left[\left(-\frac{1}{\cancel{2}}\right)\frac{\cancel{2}\cancel{s}}{(s^2+L^2)^{3/2}}\right]\hat{z} = \boxed{-\frac{\mu_0 I a n}{2\pi}\,\frac{L}{(L^2+s^2)^{3/2}}\,\hat{z}}$$

(c) In the limit $L\to\infty$, part a $\Rightarrow$ $\vec{B} = B_z\hat{z}$ everywhere, and part b $\Rightarrow$ $\vec{B}\to 0$ for $s \gg R$.




Consider two Amperian loops:
Loop (1) gives $B_z(s)L + 0 + 0 + 0 = \mu_0 I_{encl} = 0 \Rightarrow B_z(s) = 0$ for all points outside the coil. Then Loop (2) gives

$$B_z(s_{in})L + 0 + 0 + 0 = \mu_0 I_{encl} = \mu_0 I n L \Rightarrow B_z(s_{in}) = \mu_0 I n.$$

Thus,
$$\boxed{\vec{B} = \begin{cases} \mu_0 I n\,\hat{z} & \text{everywhere inside the coil} \\ 0 & \text{everywhere outside the coil} \end{cases}}$$