Physics 305 – Homework Set 10
Due submitted to eCampus no later than 5 pm on Tuesday, April 28

This is the last homework set for PHYS 305!

Do the following six problems from Griffiths:

12.35 (pg 541-42).

12.36 (pg 542).

12.41 (pg 549).

12.43 (pg 560-61).

12.48 (pg 562) Skip part (d).

> Do this problem by transforming the space-time coordinates of each point in space, while simultaneously transforming the **E** and **B** fields at that point. **Do not** do it using the Lorentz transformation properties of the wave vector that we discussed in class. (Part of the point of this problem is for you to see that you end up with the same answer!)

12.60 (pg 571).

Also do the following additional problem:

If we look out toward the sky at night, we see a distribution of stars which appears to be uniform, given by $dN/d\Omega = N_0$. Consider an observer moving past us in the $+z$ direction with a velocity $c\beta$.

(a) What is the angular distribution of stars $dN/d\Omega'$ seen by this observer, expressed in terms of N_0, β, γ, and quantities measured in her coordinate system (**not ours**)?

(b) Check your answer by showing that the moving observer sees the same total number of stars that we do.

Comment: Be careful with the "direction of observation." If we "see" a star in the direction (θ,φ), that means that the starlight is moving in the direction $(\pi-\theta,\pi+\varphi)$.

Reminder: $d\Omega = \sin(\theta)\, d\theta\, d\varphi$.

Griffiths, Problem 12.35

Classically, $\overset{m\ \ v}{\bullet\!\longrightarrow}\quad \overset{v\ \ m}{\longleftarrow\!\bullet}$ is equivalent to $\overset{m\ \ 2v}{\bullet\!\longrightarrow}\quad \overset{m}{\bullet}$ AT REST

The kinetic energy is $\frac{1}{2}mv^2$, so the relative energy (in either configuration) is four times the kinetic energy of either particle in the left-hand case.

Now consider two identical relativistic particles with 4-momenta p_1 + p_2 observed in the collider frame (where $\vec{p}_2 = -\vec{p}_1$) and the fixed-target frame where $\vec{p}_2 = 0$.

$(p_1+p_2)^2$ is Lorentz-invariant, so:

$$(p_1+p_2)^2_{\text{COLLIDER}} = -\frac{(E+E)^2}{c^2} + 0^2 = (p_1+p_2)^2_{\text{FIXED-TARGET}}$$

From $-\frac{\tilde{E}}{c} * mc + 0$

$$= p_1^2 + 2p_1 \cdot p_2 + p_2^2 = -m^2c^2 - 2\tilde{E}m - m^2c^2$$

[Nuclear and particle physicists avoid all these minus signs!]

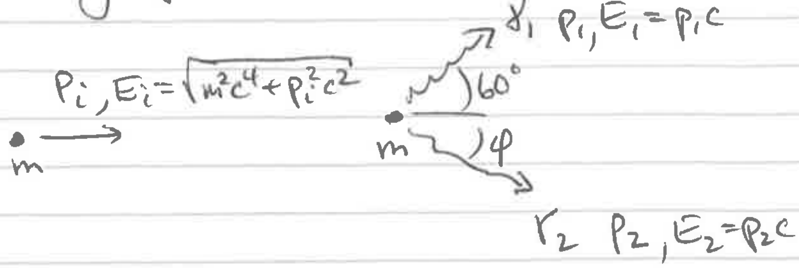
$$\Rightarrow 2\tilde{E}m = \frac{4E^2}{c^2} - 2m^2c^2 \Rightarrow \boxed{\tilde{E} = \frac{2E^2}{mc^2} - mc^2}\text{, as desired.}$$

$$\tilde{E} = \frac{2(30\text{ GeV})^2}{1\text{ GeV}} - (1\text{ GeV}) \boxed{= 1799\text{ GeV}}$$

This is approximately $\boxed{\text{60 times}}$ the two single proton energies!

Griffiths, Problem 12.35

A single photon would need to be at rest in the CM frame. Rather:



Cons of E: $\sqrt{m^2c^4 + p_i^2c^2} + mc^2 = E_1 + E_2 = p_1 c + p_2 c$

$$\Rightarrow \text{①} \quad \sqrt{m^2c^2 + p_i^2} + mc = p_1 + p_2$$

Cons of p_x: $p_i = p_1 \cos 60^\circ + p_2 \cos\varphi \Rightarrow$ ② $p_2 \cos\varphi = p_i - \frac{1}{2}p_1$

Cons of p_y: ③ $p_1 \sin 60^\circ = p_2 \sin\varphi$

②^2 + ③^2 $\Rightarrow$ $p_i^2 - p_i p_1 + \frac{1}{4}p_1^2 + \frac{3}{4}p_1^2 = p_2^2$

$$\Rightarrow \text{④} \quad p_2^2 = p_i^2 - p_i p_1 + p_1^2$$

From ①, we ~~have~~ also have: ⑤ $p_2^2 = \left[mc + \sqrt{m^2c^2 + p_i^2} - p_1\right]^2$

$$= m^2c^2 + 2mc\sqrt{m^2c^2+p_i^2} - 2mcp_1 + m^2c^2 + p_i^2 - 2\sqrt{m^2c^2+p_i^2}\,p_1 + p_1^2$$

Equating with ④ (and collecting the $\sqrt{\ }$'s together) gives:

$$\cancel{p_i^2} - p_i p_1 + \cancel{p_1^2} = 2m^2c^2 + \cancel{p_i^2} - 2mcp_1 + \cancel{p_1^2} + 2\sqrt{m^2c^2+p_i^2}\,(mc - p_1)$$

$$\Rightarrow 0 = 2m^2c^2 + 2\sqrt{m^2c^2+p_i^2}\,(mc)$$
$$- p_1\left[2mc + 2\sqrt{m^2c^2+p_i^2} - p_i\right]$$

$$\Rightarrow p_1 = \frac{m^2c^2 + \sqrt{m^2c^2+p_i^2}\,(mc)}{mc + \sqrt{m^2c^2+p_i^2} - \frac{1}{2}p_i} \Rightarrow \boxed{E_1 = mc^2\left[\frac{mc + \sqrt{m^2c^2+p_i^2}}{mc + \sqrt{m^2c^2+p_i^2} - \frac{1}{2}p_i}\right]}$$

Griffiths, Problem 12.40

For a particle moving with $\vec{u}$ under $\vec{E} + \vec{B}$ fields:

$$\vec{F} = q\vec{E} + q(\vec{u}\times\vec{B}) \overset{\text{Eq. 12.74}}{=} \frac{m}{\sqrt{1-\frac{u^2}{c^2}}}\left[\vec{a} + \frac{\vec{u}(\vec{u}\cdot\vec{a})}{c^2-u^2}\right]$$

If I dot both ~~sy~~ sides with $\vec{u}$, I get:

$$q\,\vec{u}\cdot\vec{E} = \frac{m}{\sqrt{1-\frac{u^2}{c^2}}}\left[\vec{u}\cdot\vec{a} + \frac{u^2}{c^2-u^2}\,\vec{u}\cdot\vec{a}\right]$$

$$= \frac{m}{\sqrt{1-\frac{u^2}{c^2}}}\left(\frac{c^2}{c^2-u^2}\right)\vec{u}\cdot\vec{a} \Rightarrow$$

$\vec{u}\cdot\vec{a} = \frac{q}{m}\sqrt{1-\frac{u^2}{c^2}}\left(\frac{c^2-u^2}{c^2}\right)\vec{u}\cdot\vec{E}$. Then, rearranging the above gives:

$$\vec{a} = \frac{q}{m}\sqrt{1-\frac{u^2}{c^2}}\left[\vec{E} + \vec{u}\times\vec{B}\right] - \frac{\vec{u}(\vec{u}\cdot\vec{a})}{c^2-u^2}$$

$$= \frac{q}{m}\sqrt{1-\frac{u^2}{c^2}}\left[\vec{E} + \vec{u}\times\vec{B} - \frac{\cancel{c^2-u^2}}{c^2\cancel{(c^2-u^2)}}\,\vec{u}(\vec{u}\cdot\vec{E})\right]$$

$$\Rightarrow \boxed{\vec{a} = \frac{q}{m}\sqrt{1-\frac{u^2}{c^2}}\left[\vec{E} + \vec{u}\times\vec{B} - \frac{\vec{u}(\vec{u}\cdot\vec{E})}{c^2}\right]}$$, as desired.

Griffiths, Problem 12.43

(a) $\boxed{\vec{E}_0 = \frac{\sigma_0}{\varepsilon_0}\left(\frac{\hat{y}-\hat{x}}{\sqrt{2}}\right)}$

(b) $E_x = E_{0x} = \frac{-\sigma_0}{\sqrt{2}\,\varepsilon_0}$ $\quad E_y = \gamma E_{0y} = \frac{\gamma\sigma_0}{\sqrt{2}\,\varepsilon_0}$ (since initial $B_z = 0$)

$\Rightarrow \boxed{\vec{E} = \frac{\sigma_0}{\sqrt{2}\,\varepsilon_0}\left(\gamma\hat{y}-\hat{x}\right)}$

(c) All x distances are reduced by a factor $\frac{1}{\gamma}$, while y distances are unchanged. The original slope of the planes is 1, so the final slope is γ, and they make an angle $\boxed{\theta = \tan^{-1}(\gamma)}$ with the x axis.

(d) From c, the $\perp$ to the plates is given by the direction $\frac{\gamma\hat{x}-\hat{y}}{\sqrt{1+\gamma^2}}$

For $\gamma > 1$, this is <u>not</u> parallel or anti-parallel to $\vec{E}$, so

$\vec{E}$ is <u>not</u> $\perp$ to the plates in the moving frame.

Griffiths, Problem 12.48

(a) $\boxed{\vec{E} = E_0 \hat{y} \cos(kx - \omega t) \;,\; \vec{B} = \frac{E_0}{c} \hat{z} \cos(kx-\omega t), \text{ with } k = \frac{\omega}{c}.}$

(b) $\vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp) = (E_0 \cos)(\gamma)\left(\hat{y} + \frac{v}{c}\hat{x} \times \hat{z}\right)$

$= (E_0 \cos)\gamma\left(1 - \frac{v}{c}\right)\hat{y} = \left(E_0 \sqrt{\frac{1-\beta}{1+\beta}} \cos\right)\hat{y}$

$\vec{B}'_\perp = \gamma\left(\vec{B}_\perp - \vec{\beta} \times \frac{\vec{E}_\perp}{c}\right) = (E_0 \cos)(\gamma)\left(\frac{\hat{z}}{c} - \frac{v}{c}\hat{x} \times \frac{\hat{y}}{c}\right)$

$= \left(\frac{E_0}{c}\cos\right)\gamma\left(1 - \frac{v}{c}\right)\hat{z} = \left(\frac{E_0}{c}\sqrt{\frac{1-\beta}{1+\beta}} \cos\right)\hat{z}.$

But we also must write the argument of the cos in terms of x', t'.

$ct' = \gamma(ct - \beta x)$ $\Rightarrow$ $ct = \gamma(ct' + \beta x')$

$x' = \gamma(x - \beta ct)$ $\quad x = \gamma(x' + \beta ct')$

$\Rightarrow t = \gamma\left(t' + \beta \frac{x'}{c}\right)$ and

$kx - \omega t = k[\gamma x' + \gamma\beta c t'] - \omega[\gamma(t') + \gamma\beta \frac{x'}{c}]$

$= \frac{\omega}{c}\gamma x' + \omega\gamma\beta t' - \omega\gamma t' - \frac{\omega}{c}\gamma\beta x'$

$= \frac{\omega}{c}\gamma(1-\beta)x' - \omega\gamma(1-\beta)t' = \frac{\omega}{c}\sqrt{\frac{1-\beta}{1+\beta}}\, x' - \omega\sqrt{\frac{1-\beta}{1+\beta}}\, t'$

Let $\alpha = \sqrt{\frac{1-\beta}{1+\beta}}$. Then:

$\vec{E}' = \alpha E_0 \hat{y}' \cos(\alpha k x' - \alpha\omega t')$

$\vec{B}' = \alpha \frac{E_0}{c} \hat{z}' \cos(\alpha k x' - \alpha \omega t')$

(c) $\boxed{\omega' = \alpha\omega} = \sqrt{\frac{1-\beta}{1+\beta}}\,\omega$ and $\boxed{\lambda' = \frac{2\pi}{k'} = \frac{2\pi}{\alpha k}} = \frac{2\pi c}{\alpha\omega}$

$v = \frac{\omega'}{2\pi}\lambda' = \frac{\alpha\omega}{2\pi}\frac{2\pi c}{\alpha\omega} = c$ as expected.

Griffiths, Problem 12.60

The minimum energy in the CM frame occurs when the K + Σ are produced at rest. Then $E_{cm} = 500\text{ MeV} + 1200\text{ MeV} = 1700\text{ MeV}$.

$E^2 - p^2c^2$ is both conserved and Lorentz invariant ⇒ we must have

$$E_i^2 - p_i^2c^2 = (1700\text{ MeV})^2 = E_{cm}^2$$

$$E_i^2 - p_i^2c^2 = (E_\pi + E_p)^2 - p_\pi^2c^2 = E_\pi^2 + 2E_\pi E_p + E_p^2 - p_\pi^2c^2$$

$$= m_\pi^2c^4 + 2E_\pi m_pc^2 + m_p^2c^4 = E_{cm}^2 \Rightarrow$$

$$E_\pi = \frac{E_{cm}^2 - m_\pi^2c^4 - m_p^2c^4}{2(m_pc^2)} = \frac{(1700\text{ MeV})^2 - (150\text{ MeV})^2 - (900\text{ MeV})^2}{2(900\text{ MeV})}$$

$$= 1143\text{ MeV}$$

$$\Rightarrow p_\pi = \sqrt{\frac{E_\pi^2}{c^2} - \frac{m_\pi^2c^4}{c^2}} = \frac{\sqrt{(1143\text{ MeV})^2 - (150\text{ MeV})^2}}{c} = 1133\ \frac{\text{MeV}}{c},$$

as given in the text.

<u>Note</u>: this has way too many significant digits for the mass values that were given!

SR, EP. 1

Special Relativity, Extra Problem

(a) $\frac{dN}{d\Omega'} = \frac{dN}{d\Omega}\frac{d\Omega}{d\Omega'} = \frac{dN}{d\Omega}\frac{\sin\theta\, d\theta\, d\varphi}{\sin\theta'\, d\theta'\, d\varphi'}$. $d\varphi = d\varphi' \Rightarrow$

$\frac{dN}{d\Omega'} = \frac{N_0}{\sin\theta'}\frac{-d(\cos\theta)}{d\theta'}$. If we observe a star at (θ,φ) and (θ',φ'), then following the comment and the propagation formula we found in class, we have $\tan(\pi-\theta') = \frac{\sin(\pi-\theta)}{\gamma[\cos(\pi-\theta)-\beta]}$

$\Rightarrow -\tan\theta' = \frac{\sin\theta}{\gamma(-\cos\theta-\beta)} \Rightarrow \tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta+\beta)}$ transforms $S\to S'$.

To transform $S'\to S$, we swap primed + unprimed variables and flip the sign of $\beta \Rightarrow$ $\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta'-\beta)}$. But we need $\cos\theta$, not $\tan\theta$, so:

$$\cos\theta = \frac{1}{\sec\theta} = \frac{\pm 1}{\sqrt{1+\tan^2\theta}} = \frac{\pm 1}{\sqrt{1+\frac{\sin^2\theta'}{\gamma^2(\cos\theta'-\beta)^2}}}$$

$$= \frac{\pm\gamma(\cos\theta'-\beta)}{\sqrt{\gamma^2\cos^2\theta' - 2\gamma^2\beta\cos\theta' + \gamma^2\beta^2 + \sin^2\theta'}}$$

. Note: For $0\le\theta\le\pi$, $\cos\theta$ and $\tan\theta$ have the same sign, and $\sin\theta\ge 0 \Rightarrow$ need + sign only $\rightarrow$

$$\cos\theta = \frac{\gamma(\cos\theta'-\beta)}{\sqrt{\gamma^2\cos^2\theta' - 2\gamma^2\beta\cos\theta' + \gamma^2\beta^2 + 1 - \cos^2\theta'}}$$

$$(\gamma^2-1)\cos^2\theta' = \left[\frac{1}{1-\beta^2}-1\right]\cos^2\theta' = \frac{\beta^2}{1-\beta^2}\cos^2\theta' = \gamma^2\beta^2\cos^2\theta',$$

and $\gamma^2\beta^2+1 = (\gamma^2-1)+1 = \gamma^2 \Rightarrow$

$$\cos\theta = \frac{\gamma(\cos\theta'-\beta)}{\sqrt{\gamma^2\beta^2\cos^2\theta' - 2\gamma^2\beta\cos\theta' + \gamma^2}} = \frac{\cancel{\gamma}(\cos\theta'-\beta)}{1-\beta\cos\theta'}$$ $\leftarrow$ To get correct sign!

$$\Rightarrow \frac{d(\cos\theta)}{d\theta'} = \frac{(1-\beta\cos\theta')(-\sin\theta') - (\cos\theta'-\beta)(\beta\sin\theta')}{(1-\beta\cos\theta')^2}$$

$$= -\sin\theta'\left[\frac{1-\cancel{\beta\cos\theta'}\ \cancel{+\beta\cos\theta'}-\beta^2}{(1-\beta\cos\theta')^2}\right] = \frac{-\sin\theta'}{\gamma^2(1-\beta\cos\theta')^2} \Rightarrow$$

$$\frac{dN}{d\Omega'} = \frac{N_0}{\gamma^2[1-\beta\cos\theta']^2}$$ gives the observed distribution.

Most of the stars end up in a small "hot spot" near $\theta'=0$ (i.e., in the forward direction). Science fiction movies often get this backwards.

(b) We see $\int \frac{dN}{d\Omega}d\Omega \boxed{= 4\pi N_0}$ total stars. The moving observer sees:

$$\frac{N_0}{\gamma^2}\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \frac{\sin\theta' d\theta'}{(1-\beta\cos\theta')^2} = \frac{2\pi N_0}{\gamma^2\beta}\int_0^{\pi}\frac{\beta\sin\theta'}{(1-\beta\cos\theta')^2}d\theta'$$

$$= \frac{2\pi N_0}{\gamma^2\beta}\left[\frac{-1}{1-\beta\cos\theta'}\right]_{\theta'=0}^{\pi} = \frac{2\pi N_0}{\gamma^2\beta}\left[\frac{1}{1-\beta}-\frac{1}{1+\beta}\right]$$

$$= \frac{2\pi N_0}{\gamma^2\beta}\,\frac{1+\beta-1+\beta}{1-\beta^2} = \frac{2\pi N_0}{\gamma^2\cancel{\beta}}\,\frac{2\cancel{\beta}}{1-\beta^2} = \frac{4\pi N_0}{\gamma^2(1-\beta^2)} \boxed{= 4\pi N_0}$$

$\Rightarrow$ Both observers see the same total number of stars, as desired.