Physics 305 – Sample Final Exam

There are four problems on this exam. Each problem is worth 25 points. Start each problem on a new sheet of paper, and use only one side of each sheet. GOOD LUCK !!!

(1) A particle with charge q is moving with constant speed βc along the z axis. Use the tools that we developed in Chapter 12 to determine the electric and magnetic fields at an arbitrary point in space at the time $t=0$ when the particle passes through the origin. Express your answers in cylindrical coordinates.

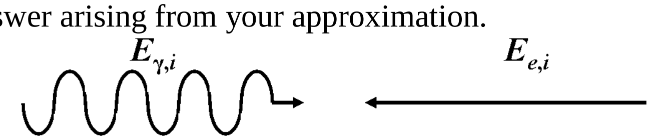
(2) Consider two thin, concentric spherical shells with radii a and b (with $a < b$). The region in between the two shells is filled with a material that has relative dielectric $\varepsilon_r = 1$, relative permeability $\mu_r = 1$, and conductivity σ. At $t=0$, there is a total charge $+Q_0$ on the inner shell and $-Q_0$ on the outer shell. Assume the conductivity is small enough so that retardation effects can be neglected.

(a) Find the electric and magnetic fields throughout space for all times $t>0$. Be sure to justify any assumptions that you make.

(b) Show that your fields from part (a) obey Maxwell's equations.

(3) The Bohr planetary model of the atom was a natural outgrowth from Rutherford's discovery of the atomic nucleus. But there was a problem! Bohr's model had an electron with charge $-e$ moving with constant speed v in a circle of radius a in the x-y plane, centered on the origin. Classically, such a system should radiate and lose energy. Calculate the intensity distribution of the emitted radiation, $dP/d\Omega$, and the average total power radiated. Note that, in the Bohr model, $v << c$, so the long-wavelength approximation is valid.
Side comment: The calculated total power isn't too far off for excited states in the hydrogen atom, but it fails miserably for the *ground* state.

(4) My former post-doc, now an Assistant Professor at James Madison University, would like to perform an experiment that requires a tunable, mono-energetic beam of 5-10 MeV gamma rays. Such a beam can be produced by shining light from a tunable dye laser operating in the IR or visible range (photon energies of <1 to a few eV) into a high energy electron beam (typical electron energies for this purpose are in the range of 0.5 to 5 GeV), as shown in the figure below. If Adriana places her target where it will see only those laser photons that backscatter off the electron beam in a narrow cone at 180^0, it will see mono-energetic gamma rays. Calculate the energy of the gamma rays, $E_{\gamma,f}$, expressed in terms of the incident photon energy $E_{\gamma,i}$, the electron beam energy $E_{e,i}$, the mass m of the electron (note: $mc^2 \approx 0.5$ MeV), and any physical constants that you need.
Note: Adriana needs to know the gamma ray energy to within ~0.1% (*i.e.*, around 10 keV). So it will be okay to make approximations that change the answer less than that. If you find you must make an approximation that has a larger impact on your final answer, also estimate the error in your answer arising from your approximation.



Final Exam, Problem 1

We want the $\vec{E}$ + $\vec{B}$ fields in S, whereas we know $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$ and $\vec{B} = 0$ in S'. First, locate the common points in 4-space:

$$z' = \gamma(z - \beta x^0) = \gamma(z - \beta c \cdot 0) = \gamma z.$$

$\vec{x}'_\perp = \vec{x}_\perp \Rightarrow y' = y$, $x' = x$. t' is not needed because $\vec{E}$ + $\vec{B}$ are constant in S'. At $(x', y', z') = (x, y, \gamma z)$:

$$E'_z = \frac{q}{4\pi\varepsilon_0}\frac{1}{s^2+(\gamma z)^2}\frac{\gamma z}{\sqrt{s^2+(\gamma z)^2}},$$

$$E'_s = \frac{q}{4\pi\varepsilon_0}\frac{1}{s^2+(\gamma z)^2}\frac{s}{\sqrt{s^2+(\gamma z)^2}}, \quad E'_\varphi = 0, \quad \vec{B}' = 0$$

Then $\vec{E}_\parallel = \vec{E}'_\parallel \Rightarrow E_z = \frac{q}{4\pi\varepsilon_0}\frac{\gamma z}{(s^2+(\gamma z)^2)^{3/2}}$

$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{v}\times\vec{B}) \Rightarrow \vec{E}_\perp = \gamma(\vec{E}'_\perp - \vec{v}\times\vec{B}') \Rightarrow$$

$$E_s = \frac{q}{4\pi\varepsilon_0}\frac{\gamma s}{(s^2+(\gamma z)^2)^{3/2}}$$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma}{(s^2+(\gamma z)^2)^{3/2}}(s\hat{s} + z\hat{z}) = \frac{q}{4\pi\varepsilon_0}\frac{\gamma\vec{r}}{(s^2+(\gamma z)^2)^{3/2}}$$

$$\vec{B}'_\perp = \gamma\left(\vec{B}_\perp - \vec{\beta}\times\frac{\vec{E}_\perp}{c}\right) \Rightarrow \vec{B}_\perp = \gamma\left(\frac{\vec{v}}{c^2}\times\vec{E}'_\perp\right) = \gamma\frac{\beta}{c}(\hat{z}\times\vec{E}'_\perp)$$

$$= \frac{q\gamma\frac{\beta}{c}}{4\pi\varepsilon_0}\frac{s}{(s^2+(\gamma z)^2)^{3/2}}\hat{\varphi}$$

Final Exam, Problem 2

(a) Let $+Q(t)$ be on the inner shell and $-Q(t)$ on the outer. Then for $a<r<b$, $\vec{E} = \frac{Q(t)}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} \Rightarrow \vec{J} = \sigma\vec{E} = \frac{\sigma Q(t)}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} \Rightarrow$

$$-\frac{dQ}{dt} = \oint \vec{J}\cdot d\vec{a} = \frac{\sigma Q(t)}{\cancel{4\pi}\varepsilon_0}\frac{\cancel{4\pi}\cancel{r^2}}{\cancel{r^2}} = \frac{\sigma Q(t)}{\varepsilon_0} \Rightarrow$$

$$\frac{dQ}{dt} + \frac{\sigma}{\varepsilon_0}Q = 0 \Rightarrow \boxed{Q(t) = Q_0 e^{-\frac{\sigma}{\varepsilon_0}t}}$$ Thus,

$$\boxed{\vec{E}(t) = \frac{Q_0}{4\pi\varepsilon_0} e^{-\frac{\sigma}{\varepsilon_0}t}\frac{\hat{r}}{r^2} \text{ for } a<r<b\text{, and zero elsewhere}}.$$

The current is spherically ~~sy~~ symmetric, so no angular direction differs from any other. Thus, we must have $\vec{B} = B_r\hat{r}$. But $\vec{\nabla}\cdot\vec{B} = 0$ $\Rightarrow \oint\vec{B}\cdot d\vec{a} = 0$. Considering a sphere of radius r then gives:

$$\boxed{\vec{B}(t) = 0 \text{ for } \underline{\underline{\text{all}}}\ r}.$$

(b) $\vec{E}$ is just the electrostatic solution for $Q(t)$ at time t, so it obeys Gauss' Law, and has $\vec{\nabla}\times\vec{E} = 0 = -\frac{\partial\vec{B}}{\partial t}$, so Faraday's law is okay.

By construction, we have assured that $\vec{\nabla}\cdot\vec{B} = 0$.

Finally, Maxwell-Ampere's law gives:

$$\vec{\nabla}\times\vec{B} = 0 = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\sigma\frac{Q(t)}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} + \mu_0\cancel{\varepsilon_0}\frac{1}{4\pi\cancel{\varepsilon_0}}\left(\frac{\partial Q(t)}{\partial t}\right)\frac{\hat{r}}{r^2}$$

$$= \frac{\mu_0\sigma}{4\pi\varepsilon_0}Q(t)\frac{\hat{r}}{r^2} + \frac{\mu_0}{4\pi}\left(-\frac{\sigma}{\varepsilon_0}Q(t)\right)\frac{\hat{r}}{r^2} = 0 \checkmark,$$

and Maxwell-Ampere's law is also okay.

Final Exam, Problem 3

Assume $t=0$ is chosen when the electron is on the x-axis. then:

$\vec{w}(t) = a\cos\left(\frac{vt}{a}\right)\hat{x} + a\sin\left(\frac{vt}{a}\right)\hat{y}$. Thus,

$\vec{p}(t) = \int r'\rho(t)\,d\tau = -ea\left(\cos\left(\frac{vt}{a}\right)\hat{x} + \sin\left(\frac{vt}{a}\right)\hat{y}\right)$, and

$\ddot{\vec{p}}(t) = e\cancel{a}\frac{v^2}{a^{\cancel{2}}}\left[\cos\left(\frac{vt}{a}\right)\hat{x} + \sin\left(\frac{vt}{a}\right)\hat{y}\right]$.

$|\ddot{\vec{p}}| = \frac{ev^2}{a}$, independent of time, so the time-averaged total radiated power is given by:

$$P = \frac{\mu_0 \ddot{p}^2}{6\pi c} = \boxed{\frac{\mu_0 e^2 v^4}{6\pi a^2 c}}.$$

For the angular distribution, $\langle\cos^2\rangle = \langle\sin^2\rangle = \frac{1}{2}$, while $\langle\sin\cos\rangle = 0$

Therefore, we can think of this as two independent, harmonic oscillators, one in $\hat{x}$ and one in $\hat{y}$, that add incoherently. Each radiates at $\sin^2\theta$ relative to the oscillation direction. The direction cosine relative to x is $\sin\theta\cos\varphi \Rightarrow$

$\sin^2\theta_x = 1-\cos^2\theta_x = 1-\sin^2\theta\cos^2\varphi$. Likewise,

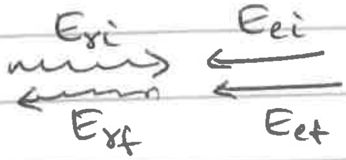
$\sin^2\theta_y = 1-\cos^2\theta_y = 1-\sin^2\theta\sin^2\varphi$, and

$$\left\langle\frac{dP}{d\Omega}\right\rangle = r^2\langle\vec{S}\rangle\cdot\hat{r} = \frac{\mu_0 e^2 v^4}{32\pi^2 c a^2}\left[1-\sin^2\theta\cos^2\varphi + 1-\sin^2\theta\sin^2\varphi\right]$$

$$= \frac{\mu_0 e^2 v^4}{32\pi^2 c a^2}\left(2-\sin^2\theta\right) = \boxed{\frac{\mu_0 e^2 v^4}{32\pi^2 c a^2}\left(1+\cos^2\theta\right)}.$$

A quick check shows $\int\frac{dP}{d\Omega}d\Omega = P_{tot}$, as it must.

Final Exam, Problem 4



Cons of p^μ $\Rightarrow$ $E_{\gamma i} + E_{ei} = E_{\gamma f} + E_{ef}$

$p_{ei} - p_{\gamma i} = p_{ef} + p_{\gamma f}$

$$\Rightarrow \sqrt{\frac{E_{ei}^2}{c^2} - m^2c^2} - \frac{E_{\gamma i}}{c} = \sqrt{\frac{E_{ef}^2}{c^2} - m^2c^2} + \frac{E_{\gamma f}}{c}$$

$$\Rightarrow E_{ei}\left(1 - \frac{1}{2}\frac{m^2c^4}{E_{ei}^2}\right) - E_{\gamma i} \approx E_{ef}\left(1 - \frac{1}{2}\frac{m^2c^4}{E_{ef}^2}\right) + E_{\gamma f}$$

$$\Rightarrow E_{ei} - \frac{1}{2}\frac{m^2c^4}{E_{ei}} - E_{\gamma i} = E_{ef} - \frac{1}{2}\frac{m^2c^4}{E_{ef}} + E_{\gamma f}$$

$$\Rightarrow -\frac{1}{2}\frac{m^2c^4}{E_{ei}} - E_{\gamma i} = E_{\gamma i} - \frac{1}{2}\frac{m^2c^4}{E_{ef}}$$

$$\Rightarrow 4E_{\gamma i} + \frac{m^2c^4}{E_{ei}} = \frac{m^2c^4}{E_{ef}} = \frac{m^2c^4}{E_{\gamma i} + E_{ei} - E_{\gamma f}}$$

$E_{\gamma i} \ll 10$ keV, so it can be discarded in the denominator. Then:

$$E_{ei} - E_{\gamma f} = \frac{m^2c^4}{4E_{\gamma i} + \frac{m^2c^4}{E_{ei}}} = \frac{E_{ei}}{1 + 4\frac{E_{\gamma i}E_{ei}}{m^2c^4}}$$

$$\Rightarrow \cancel{E_{ei}} + 4\frac{E_{\gamma i}E_{ei}^2}{m^2c^4} - E_{\gamma f}\left(1 + 4\frac{E_{\gamma i}E_{ei}}{m^2c^4}\right) = \cancel{E_{ei}}$$

$$\Rightarrow \boxed{E_{\gamma f} = \frac{4\frac{E_{\gamma i}E_{ei}^2}{m^2c^4}}{1 + 4\frac{E_{\gamma i}E_{ei}}{m^2c^4}} = \frac{4E_{\gamma i}E_{ei}^2}{m^2c^4 + 4E_{\gamma i}E_{ei}}}$$