Physics 305 – Sample Exam 2

There are four problems on this exam. Each problem is worth 25 points. Start each problem on a new sheet of paper, and use only one side of each sheet. GOOD LUCK !!!

(1) Assume you have two uniform, isotropic media, separated by a planar interface, with indices of refraction n_1 and n_2. ($\mu_1 = \mu_2 = \mu_0$.) A plane electromagnetic wave propagating in medium 1 strikes the interface at normal incidence.

a) Derive expressions for the reflection and transmission coefficients. (Note: If you simply copy expressions from your crib sheet, you will receive zero credit for this part.)

b) What condition(s) do n_1 and n_2 need to obey to obtain $T = R$?

(2) A point charge q moves in a circle of radius a in the x-y plane, centered on the origin, with constant angular speed ω. To be specific, assume its location at any given time is given by:

$$a\cos(\omega t)\hat{x} + a\sin(\omega t)\hat{y}$$

Find the Lienard-Wiechert potentials for points on the z axis.

(3) A point charge q is located above a horizontal, perfect conducting plane. The conductor forms the x-y plane, and the point charge at time t is on the $+z$ axis at the location

$$z = a + b\cos(\omega t).$$

Assume a and b are both $\ll c/\omega$.

a) What is the angular distribution of the emitted radiation?

b) What is the total radiated power?

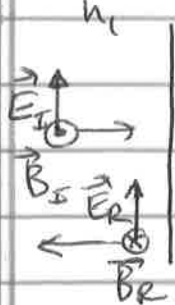
(4) A hollow rectangular waveguide with perfect conducting walls has inner dimensions a and b. To be specific, assume the walls are on the planes x=0, x=a, y=0, and y=b, and a>b.

a) Find $E_z(x,y)$ for TM modes.

b) What is the lowest cut-off frequency for a TM mode?

Exam 2, Problem 1

(a)



$\vec{E}_\parallel$ + $\vec{H}_\parallel$ must be continuous at the interface ⇒

① $E_I + E_R = E_T$ $\qquad \frac{B_I}{\mu_1} - \frac{B_R}{\mu_1} = \frac{B_T}{\mu_2}$

with $B_I = \frac{1}{v_1}E_I$, $B_R = \frac{1}{v_1}E_R$, $B_T = \frac{1}{v_2}E_T$ and $\mu_1 = \mu_2$ ⇒

② $E_I - E_R = \frac{v_1}{v_2}E_T$. ①+② ⇒ $2E_I = \left(1+\frac{v_1}{v_2}\right)E_T$ ⇒

$$E_T = \frac{2v_2}{v_1+v_2}E_I = \frac{2\frac{c}{n_2}}{\frac{c}{n_1}+\frac{c}{n_2}}E_I = \boxed{\frac{2n_1}{n_1+n_2}E_I}$$

$\frac{v_1}{v_2}$① − ② ⇒ $\left(\frac{v_1}{v_2}-1\right)E_I + \left(\frac{v_1}{v_2}+1\right)E_R = 0$ ⇒

$$E_R = \frac{v_2-v_1}{v_2+v_1}E_I = \frac{\frac{c}{n_2}-\frac{c}{n_1}}{\frac{c}{n_2}+\frac{c}{n_1}}E_I = \boxed{\frac{n_1-n_2}{n_1+n_2}E_I}$$

$$R = \left|\frac{E_R}{E_I}\right|^2 = \frac{(n_1-n_2)^2}{(n_1+n_2)^2} \qquad T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left|\frac{E_T}{E_I}\right|^2 = \frac{n_2^2\frac{c}{n_2}}{n_1^2\frac{c}{n_1}}\frac{4n_1^2}{(n_1+n_2)^2}$$

$$= \frac{n_2}{\cancel{n_1}}\frac{4n_1^{\cancel{2}}}{(n_1+n_2)^2} \Rightarrow T = \frac{4n_1n_2}{(n_1+n_2)^2}$$

(b) $T=R$ requires $(n_1-n_2)^2 = 4n_1n_2$ ⇒ $n_1^2 - 2n_1n_2 + n_2^2 = 4n_1n_2$ ⇒

$$n_2^2 - 6n_1n_2 + n_1^2 = 0 \Rightarrow n_2 = \frac{6n_1 \pm\sqrt{36n_1^2 - 4n_1^2}}{2} = (3\pm\sqrt{8})n_1$$

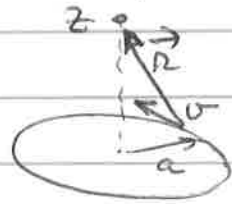
$$\Rightarrow n_2 = (3\pm\sqrt{8})n_1$$

Exam 2, Problem 2

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \frac{qc}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})_{ret}}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{qc\vec{v}_{ret}}{(\mathscr{r}c - \vec{\mathscr{r}}\cdot\vec{v})_{ret}} = \frac{\vec{v}_{ret}}{c^2} V(\vec{r},t)$$

In the present case, we have $\vec{\mathscr{r}} \perp \vec{v}$, so the $\vec{\mathscr{r}}\cdot\vec{v}$ terms $\to 0$.



$|\vec{\mathscr{r}}| = \sqrt{z^2+a^2}$ is constant, so:

$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \frac{q\cancel{c}}{\sqrt{z^2+a^2}\,\cancel{c}} = \boxed{\frac{q}{4\pi\varepsilon_0} \frac{1}{\sqrt{z^2+a^2}}}, \text{ indep of } t.$$

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{q\cancel{c}\vec{v}_{ret}}{\mathscr{r}\cancel{c}} = \frac{\mu_0 q}{4\pi} \frac{1}{\sqrt{z^2+a^2}} \vec{v}_{ret},$$

$$t_r = t - \frac{\mathscr{r}}{c} = t - \frac{\sqrt{z^2+a^2}}{c} \Rightarrow$$

$$\vec{v}_{ret} = -a\omega \sin(\omega t_r)\hat{x} + a\omega\cos(\omega t_r)\hat{y}, \text{ and:}$$

$$\boxed{\vec{A} = \frac{\mu_0 q\omega}{4\pi} \frac{a}{\sqrt{z^2+a^2}} \left\{ -\sin\left(\omega\left(t - \frac{\sqrt{z^2+a^2}}{c}\right)\right)\hat{x} + \cos\left(\omega\left(t - \frac{\sqrt{z^2+a^2}}{c}\right)\right)\hat{y} \right\}}$$

Exam 2, Problem 3

(a) q produces an image charge $-q$ at $-z$. $a+b$ are both $\ll \frac{c}{\omega}$, so we can ignore retardation in the formation of the image charge

Thus, $\vec{p}(t) = \int \vec{r}'\rho(\vec{r}')\,d\tau' = 2q\left(a + b\cos(\omega t)\right)\hat{z} \Rightarrow$

$\ddot{\vec{p}} = -2qb\omega^2\cos(\omega t)\hat{z}$ and $\langle \ddot{p}^2 \rangle = \overset{2}{\cancel{4}}q^2b^2\omega^4\left(\frac{1}{\cancel{2}}\right)$

$\vec{I} = \langle \vec{S} \rangle = \frac{\mu_0}{16\pi^2 c}\langle \ddot{p}^2 \rangle \left(\frac{\sin^2\theta}{r^2}\right)\hat{r} = \frac{\mu_0 q^2 b^2 \omega^4}{8\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\hat{r} \Rightarrow$

$$\frac{dP}{d\Omega} = r^2\vec{I}\cdot\hat{r} = \begin{cases} \frac{\mu_0 q^2 b^2 \omega^4}{8\pi^2 c}\sin^2\theta & \text{for } 0 \le \theta \le \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} < \theta \end{cases}$$ (no radiation below the plane)

(b) $$P = \int \frac{dP}{d\Omega}\,d\Omega = \frac{\mu_0 q^2 b^2 \omega^4}{8\pi^2 c}\underbrace{\int_0^{2\pi} d\varphi}_{=2\pi}\ \underbrace{\int_0^{\frac{\pi}{2}} \sin^2\theta\, d(\cos\theta)}_{=\int_0^1 (1-x^2)\,dx = \frac{2}{3}}$$

$$\Rightarrow P = \frac{\mu_0 q^2 b^2 \omega^4}{\underset{2}{\cancel{8}}\pi^{\cancel{2}} c}(\cancel{2\pi})\left(\frac{\cancel{2}}{3}\right) = \frac{\mu_0 q^2 b^2 \omega^4}{6\pi c}$$

Exam 2, Problem 4

(a) E_z must obey $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c^2}\right)-k^2\right]E_z=0$, while $B_z=0$.

Separation of variables gives

$$E_z=\left(A\cos(k_x x)+B\sin(k_x x)\right)\left(C\cos(k_y y)+D\sin(k_y y)\right).$$

We must have $E_z(x=0)=E_z(x=a)=E_z(y=0)=E_z(y=b)=0$ (continuity of $E_\parallel$ at the surfaces) $\Rightarrow$ $A+C=0$, $k_x=\frac{m\pi}{a}$, $k_y=\frac{n\pi}{b}$

$\Rightarrow \boxed{E_z=E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}$, with m+n positive integers.

Does this obey our b.c. on E_x, E_y, B_x, B_y?

$$E_x=\frac{ik}{\left(\frac{\omega}{c}\right)^2-k^2}\frac{\partial E_z}{\partial x}\propto\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

This is zero at $y=0$ and $y=b$, as it must be. ✓

$B_y\propto E_x$, so it is also zero at $y=0$ and $y=b$, as it must be. ✓

$$E_y=\frac{ik}{\left(\frac{\omega}{c}\right)^2-k^2}\frac{\partial E_z}{\partial y}\propto\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$ and $B_x\propto E_y$.

Again, E_y + B_x are zero at $x=0$ and $x=a$, as they must be. ✓

Works! Above is the most general TM mode.

(b) For lowest cut-off frequency, $k=0$ and ω_{mn} is a minimum.
Clearly, $m=n=1$, and $\omega_{11}=c\pi\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$