Physics 305 – Sample Exam 1 Questions

There are four problems on this exam. Each problem is worth 25 points. Start each problem on a new sheet of paper, and use only one side of each sheet. GOOD LUCK !!!

(1) An infinitely long cylinder of radius R is centered on the z axis. It carries a magnetization $\mathbf{M} = ks^3\,\hat{\boldsymbol{\varphi}}$, where k is a constant.

a) Locate all of the bound current.

b) Find the magnetic field due to $\mathbf{M}$ for points inside and outside the cylinder.

(1) A long, straight, circular cable carries a current in one direction uniformly distributed over its cross section. The current returns along the surface of the cable. (There is a very thin insulating layer separating the currents.) Find the self-inductance per unit length.

(2) A capacitor consists of two circular plates of radius a separated by a distance $d \ll a$. The capacitor is being charged using very thin wires that are connected to the (outside) centers of the circular plates and carry a constant current I. Assume the current flows out over the plates in such a way that the surface charge is always uniform, and is zero at $t = 0$. Neglect edge effects.

a) Find the displacement current through the circle of radius $s < a$ that is centered on the axis of the capacitor and in the plane midway between the plates.

b) Show that your result from part (a) exactly equals the net current that is flowing onto the plates inside the circle of radius s. (This shows that the displacement current has just the right magnitude to complete the circuit across the capacitor.)

(3) Concentric spherical shells with radii a and b ($a < b$) carry uniformly distributed charges $+Q$ (at a) and $-Q$ (at b). They are immersed in a uniform magnetic field with magnitude B_0 pointing in the z-direction. Calculate the angular momentum in the fields relative to the center.

Note: The funny numbering arises because the "first" question (1) comes from an old PHYS 304 Final Exam. Meanwhile, this overall exam is slightly shorter and easier than I usually target because this particular PHYS 304 question is shorter and easier than the original PHYS 305 question that involved material from early in Chapter 9.

Final Exam, Problem 1

a) $\vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\, ks^3\right)\hat{z} = \frac{1}{s}\left(4ks^3\right)\hat{z} = \boxed{4ks^2\hat{z}}$

$\vec{K}_b = \vec{M} \times \hat{n} = ks^3\hat{\varphi} \times \hat{s} = ks^3\left(-\hat{z}\right) = \boxed{-kR^3\hat{z}}$

Check: $\int_0^{2\pi} d\varphi \int_0^R s\,ds\, \vec{J}_b = 2\pi \left[ks^4\right]_{s=0}^{R} \hat{z} = 2\pi k R^4 \hat{z}$

$2\pi R \vec{K}_b = -2\pi k R^4 \hat{z}$, so the total bound current (volume + surface) $= 0$ ✓

b) $\vec{J} + \vec{K} \parallel \hat{z} \Rightarrow \vec{B} \perp \hat{s}$. A rotation of the coordinate system about the x- or y-axis ~~about~~ shows $B_s = 0 \Rightarrow \vec{B} = B_\varphi \hat{\varphi}$, both inside and outside. So we can use the integral form of Ampère's law:

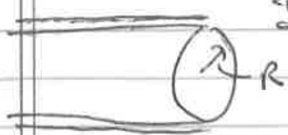
same int as above!

Inside: $\oint \vec{B} \cdot d\vec{l} = B_\varphi(s)\, \cancel{2\pi s} = \mu_0 I_{encl} = \mu_0 \cancel{2\pi} k\, s^{\cancel{4}\, 3}$

$\Rightarrow B_\varphi(s) = \mu_0 k s^3 \Rightarrow \boxed{\vec{B} = \mu_0 k s^3 \hat{\varphi}} \quad \left(= \mu_0 \vec{M}\right)$

Outside: $I_{encl} = 0 \Rightarrow B_\varphi(s) = 0 \Rightarrow \boxed{\vec{B} = 0}$

Exam 1, Problem 1



In the conductor, let the current density be $\vec{J}$, with $J = \frac{I}{\pi R^2}$. Then $\oint \vec{B}\cdot d\vec{l} = B(s)\,2\cancel{\pi}\cancel{s} = \mu_0 I_{encl} = \mu_0 J \cancel{\pi} s^{\cancel{2}}$

$$\Rightarrow B(s) = \frac{\mu_0 J}{2} s = \frac{\mu_0 I}{2\pi R^2} s.$$

$$W = \frac{1}{2\mu_0}\int B^2 d\tau = \frac{1}{2\cancel{\mu_0}} \frac{\mu_0^{\cancel{2}} I^2}{4\pi^2 R^4} \int_0^R s^2 s\,ds \int_0^{2\pi} d\varphi \times \underbrace{l}_{\text{length}}$$

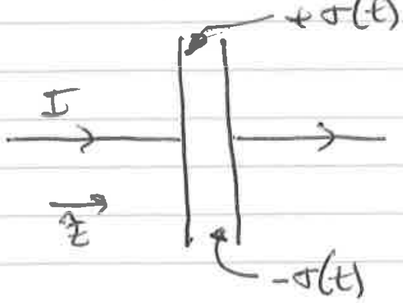
$$= \frac{1}{\cancel{2}} \frac{\mu_0 I^2}{4\pi^2 \cancel{R^4}} \frac{\cancel{R^4}}{4} \cancel{2\pi}\, l = \frac{\mu_0 I^2 l}{16\pi}$$

But $W = \frac{1}{2} L I^2 \Rightarrow$ The self-inductance per unit length is:

$\boxed{\mathcal{L} = \frac{\mu_0}{8\pi}}$, independent of R (!).

Exam 1, Problem 2

a)



The charge on the plates is $Q(t) = It \Rightarrow$

$$\sigma(t) = \frac{Q(t)}{\pi a^2} = \frac{I}{\pi a^2} t \Rightarrow$$

$$\vec{E}(t) = \frac{\sigma(t)}{\varepsilon_0}\hat{z} = \frac{I}{\pi a^2 \varepsilon_0} t\hat{z} \Rightarrow$$

$$I_{disp} = \int \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a} = \cancel{\varepsilon_0} \frac{I}{\cancel{\pi} a^2 \cancel{\varepsilon_0}} \cancel{\pi} s^2 = I \frac{s^2}{a^2}$$

b) The current I is spreading uniformly over the plates. Thus, the current being deposited in the circle of radius s is $I \frac{\pi s^2}{\pi a^2} = I \frac{s^2}{a^2}$. This equals the result from part (a), as desired.

Exam 1, Problem 3

$\vec{E} = 0$ except for $a<r<b$, where $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$.

$\vec{l}_{EM} = \varepsilon_0 \vec{r}\times(\vec{E}\times\vec{B}) \Rightarrow$ can only be non-zero for $a<r<b$.

$$\varepsilon_0 \vec{E}\times\vec{B} = \cancel{\varepsilon_0}\left(\frac{1}{4\pi\cancel{\varepsilon_0}}\frac{Q}{r^2}\hat{r}\right)\times(B_0\hat{z}) = \frac{QB_0}{4\pi r^2}\hat{r}\times\hat{z}$$

$\hat{r}\times\hat{z} = -\sin\theta\,\hat{\varphi} \Rightarrow$

$$\boxed{\varepsilon_0\vec{E}\times\vec{B} = \frac{-QB_0}{4\pi r^2}\sin\theta\,\hat{\varphi}}$$

Side note: This is the momentum density. It points in the $-\hat{\varphi}$ direction, so we should expect $\vec{L}$ will point in the $-\hat{z}$-direction.

$$\vec{l}_{EM} = \frac{-QB_0}{4\pi r^2}\sin\theta\,\vec{r}\times\hat{\varphi} = \frac{-QB_0}{4\pi r}\sin\theta\,\hat{r}\times\hat{\varphi}$$

$\hat{r}\times\hat{\varphi} = -\hat{\theta} \Rightarrow \boxed{\vec{l}_{EM} = \frac{QB_0}{4\pi r}\sin\theta\,\hat{\theta}}$

When we integrate over φ, the x- and y-components of $\vec{L}$ will cancel.

So let's focus on L_z. $\hat{z}\cdot\hat{\theta} = -\sin\theta \Rightarrow$

$$L_z = \frac{-QB_0}{4\pi}\int_a^b \frac{r^2dr}{r}\int_0^\pi \sin^2\theta\,\sin\theta\,d\theta\int_0^{2\pi}d\varphi$$

$$= -\frac{QB_0}{\cancel{4\pi}_{\,2}}\left(\frac{b^2-a^2}{2}\right)\left(\int_{-1}^{1}(1-x^2)\,dx\right)\cancel{2\pi}$$

$$= \frac{-QB_0(b^2-a^2)}{\cancel{4}}\,\frac{\cancel{4}}{3} \Rightarrow \boxed{\vec{L}_{tot,EM} = \frac{-QB_0}{3}(b^2-a^2)\hat{z}}$$