PHYS 606 – Spring 2016 – Homework VII – Solution

Problem [1]

(a) $E > V_0 \Rightarrow$ plane waves everywhere. General solution

$$\psi(x) = \begin{cases} A e^{ikx} + B e^{-ikx} & \text{f. } x < -a \\ C e^{ik'x} + D e^{-ik'x} & \text{f. } -a < x < a \\ E e^{ikx} + F e^{-ikx} & \text{f. } x > a \end{cases} \qquad k = \frac{1}{\hbar}\sqrt{2mE}, \quad k' = \frac{1}{\hbar}\sqrt{2m(E-V_0)}$$

Matching: @ $x=-a$:

$$A e^{-ika} + B e^{+ika} = C e^{-ik'a} + D e^{+ik'a} \qquad (1)$$

$$ik(A e^{-ika} - B e^{+ika}) = ik'(C e^{-ik'a} - D e^{+ik'a}) \qquad (2)$$

$$\Rightarrow ik(1)+(2):\quad A = \frac{1}{2k} e^{ika}\left[C(k+k')e^{-ik'a} + D(k-k')e^{ik'a}\right] \qquad (1')$$

$$ik(1)-(2):\quad B = \frac{1}{2k} e^{-ika}\left[C(k-k')e^{-ik'a} + D(k+k')e^{ik'a}\right] \qquad (2')$$

@ $x=+a$:

$$C e^{ik'a} + D e^{-ik'a} = E e^{ika} + F e^{-ika} \qquad (3)$$

$$ik'(C e^{ik'a} - D e^{-ik'a}) = ik(E e^{ika} - F e^{-ika}) \qquad (4)$$

$$\Rightarrow C = \frac{1}{2k'} e^{-ik'a}\left[E(k'+k)e^{ika} + F(k'-k)e^{-ika}\right] \qquad (3')$$

$$D = \frac{1}{2k'} e^{ik'a}\left[E(k'-k)e^{ika} + F(k'+k)e^{-ika}\right] \qquad (4')$$

Eliminate C, D from (1') through (4'):

$$A = \frac{1}{4kk'} E\left[(k+k')^2 e^{2i(k-k')a} - (k-k')^2 e^{2i(k+k')a}\right]$$

$$+ \frac{1}{4kk'} F\left[(k'^2-k^2)e^{-2ik'a} - (k'^2-k^2)e^{2ik'a}\right]$$

$$= E e^{2ika}\left[\frac{k^2+k'^2}{4kk'} 2i\sin(-2k'a) + \frac{2kk'}{4kk'} 2\cos(-2k'a)\right]$$

$$+ F \frac{k'^2-k^2}{4kk'} 2i\sin(-2k'a)$$

$$B = \frac{i}{4kk'} E\left[(k^2-k'^2)e^{-2ik'a} - (k^2-k'^2)e^{+2ik'a}\right] + \frac{1}{4kk'} F\left[-(k-k')^2 e^{-2i(k+k')a} + (k+k')^2 e^{2i(k'-k)a}\right]$$

$$= E\,\frac{k^2-k'^2}{4kk'}\, 2i\sin(-2k'a) + F e^{-2ika}\left[\frac{k^2+k'^2}{4kk'}\, 2i\sin 2k'a + \frac{2kk'}{4kk'}\, 2\cos 2k'a\right]$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = M \begin{pmatrix} E \\ F \end{pmatrix}$$ with M-matrix with $\varepsilon' = \frac{k'}{k} + \frac{k}{k'} = \frac{k^2+k'^2}{kk'}$

$\eta' = \frac{k'}{k} - \frac{k}{k'} = \frac{k'^2-k^2}{kk'}$

$$M = \begin{pmatrix} \left(\cos 2k'a - i\frac{\varepsilon'}{2}\sin 2k'a\right)e^{2ika} & -\frac{i\eta'}{2}\sin 2k'a \\ \frac{i\eta'}{2}\sin 2k'a & \left(\cos 2k'a + i\frac{\varepsilon'}{2}\sin 2k'a\right)e^{-2ika} \end{pmatrix}$$

$$T = \left|\frac{E}{A}\right|^2 = |M_{11}|^2 = \cos^2 2k'a + \frac{\varepsilon'^2}{4}\sin^2 2k'a \quad , \quad R = 1 - T$$

$T \to 1$ for $k' \to k$, i.e. $E \gg V_0$

(b) Situation obviously exactly the same as (a) just replace

$k' = \frac{1}{\hbar}\sqrt{2m(E+V_0)}$.

Problem [2]

(a) $$W^*(\vec{r},\vec{p},t) = \left(\int \psi^*(\vec{r}-\tfrac{\vec{r}'}{2})\,\psi(\vec{r}+\tfrac{\vec{r}'}{2})\, e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}'}\, d^3r'\right)^* \frac{1}{(2\pi\hbar)^3}$$

$$= \int \psi^*(\vec{r}+\tfrac{\vec{r}'}{2})\,\psi(\vec{r}-\tfrac{\vec{r}'}{2})\, e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}'}\, d^3r' \frac{1}{(2\pi\hbar)^3}$$

$$\underset{\vec{r}'=-\tilde{\vec{r}}}{=} \int \psi^*(\vec{r}-\tfrac{\tilde{\vec{r}}}{2})\,\psi(\vec{r}+\tfrac{\tilde{\vec{r}}}{2})\, e^{-\frac{i}{\hbar}\vec{p}\cdot\tilde{\vec{r}}}\, d^3\tilde{r}\,\frac{1}{(2\pi\hbar)^3} = W(\vec{r},\vec{p},t)$$

(b) $$W^2 = \frac{1}{(2\pi\hbar)^6}\left|\int \psi^*(\vec{r}-\tfrac{\vec{r}'}{2})\,\psi(\vec{r}+\tfrac{\vec{r}'}{2})\, e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}'}\, d^3r'\right|^2$$

$$\underset{\text{Schwartz}}{\leq} \frac{1}{(2\pi\hbar)^6} \underbrace{\int |\psi(\vec{r}-\tfrac{\vec{r}'}{2})|^2 d^3r'}_{2^3} \underbrace{\int |\psi(\vec{r}+\tfrac{\vec{r}'}{2})|^2 d^3r'}_{2^3}$$

$$= \left(\frac{2}{h}\right)^6$$

(c) $$\int W(\vec{r},\vec{p},t)\, W'(\vec{r},\vec{p},t)\, d^3r\, d^3p$$

$$= \frac{1}{(2\pi\hbar)^6}\int d^3r\, d^3p\, d^3r'\, d^3r''\; \psi^*(\vec{r}-\tfrac{\vec{r}'}{2})\,\psi(\vec{r}+\tfrac{\vec{r}'}{2})\,\psi'^*(\vec{r}-\tfrac{\vec{r}''}{2})\,\psi'(\vec{r}+\tfrac{\vec{r}''}{2})\; e^{-\frac{i}{\hbar}\vec{p}\cdot(\vec{r}'+\vec{r}'')}$$

$$= \frac{1}{(2\pi\hbar)^3}\int d^3r\, d^3r'\; \psi^*(\vec{r}-\tfrac{\vec{r}'}{2})\,\psi(\vec{r}+\tfrac{\vec{r}'}{2})\,\psi'^*(\vec{r}+\tfrac{\vec{r}'}{2})\,\psi'(\vec{r}-\tfrac{\vec{r}'}{2})$$

$$\underset{\substack{\hat{r}=\vec{r}-\frac{\vec{r}'}{2}\\ \hat{\hat{r}}=\vec{r}+\frac{\vec{r}'}{2}}}{=} \frac{1}{(2\pi\hbar)^3}\int d^3\hat{r}\; \psi^*(\hat{\vec{r}})\,\psi'(\hat{\vec{r}}) \int d^3\hat{\hat{r}}\; \psi'^*(\hat{\hat{\vec{r}}})\,\psi(\hat{\hat{\vec{r}}}) = \frac{1}{(2\pi\hbar)^3}\left|\langle\psi'|\psi\rangle\right|^2$$

Problem [3]

Let $\varphi(x)$ be an extremum of $S[\varphi]$ and $\varphi(x,\alpha) = \varphi(x) + \alpha\,\eta(x)$

a 1-parameter curve through it with $\eta(x) = 0$ on the boundary $\partial\Gamma$

$\varphi(x,\alpha)$ for small α then is a variation around $\varphi(x)$ and <u>any</u>

allowed variation can be written as an $\varphi(x,\alpha)$ with some suitable $\eta(x)$.

Then for variation $\varphi(x,\alpha)$

$$\delta S = \frac{\partial S}{\partial \alpha}\,\delta\alpha = \int_\Gamma \left(\frac{\partial \mathcal{L}}{\partial \varphi} \frac{\partial \varphi}{\partial \alpha} + \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x_i}\right)} \underbrace{\frac{\partial\left(\frac{\partial \varphi}{\partial x_j}\right)}{\partial \alpha}}_{= \frac{\partial}{\partial x_j} \frac{\partial \varphi}{\partial \alpha}} \right) d^N x \;\delta\alpha$$

$$\overset{\text{part. int.}}{=} \int_\Gamma \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x_j}\right)} \right) \underbrace{\frac{\partial \varphi}{\partial \alpha}}_{\eta(x)} \,\delta\alpha \; dx \;+\; \cancel{\text{boundary term}}$$

$\left(\frac{\partial \varphi}{\partial \alpha} = 0 \text{ on } \partial\Gamma \text{ since } \eta = 0 \text{ on } \partial\Gamma\right)$

Thus $\frac{\partial \mathcal{L}}{\partial \varphi} - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x_j}\right)} = 0 \;\Rightarrow\; \delta S = 0$

Conversely, if $\delta S = 0$ for <u>any</u> allowed choice of $\eta(x)$ then $\frac{\partial \mathcal{L}}{\partial \varphi} - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi}{\partial x_j}\right)} = 0$