PHYS 606 – Spring 2017 – Homework VI – Solution

Problem [1]

(a) Use power series

$$\frac{dH_n}{d\xi} \underset{n\text{ even}}{=} \frac{d}{d\xi}\left[(-1)^{\frac{n}{2}} \frac{n!}{(\frac{n}{2})!}\left(1 - \frac{2n}{2!}\xi^2 + \frac{2^2 n(n-2)}{4!} - \dots\right)\right]$$

$$= (-1)^{\frac{n}{2}} \frac{n!}{(\frac{n}{2})!}\left(-\frac{2n}{1!}\xi + \frac{2^2 n(n-2)}{3!}\xi^3 - \dots\right)$$

$$= -2n(-1)^{\frac{n}{2}} \frac{n}{\frac{n}{2}} \frac{(n-1)!}{(\frac{n}{2}-1)!}\left(\xi - \frac{2(n-2)}{3!}\xi^3 + \dots\right)$$

$$= 2n\,(-1)^{\frac{n-2}{2}} \frac{2(n-1)!}{(\frac{n-2}{2})!}\left(\xi - \frac{2(n-2)}{3!}\xi^3 + \dots\right) = 2n\, \underset{(n-1)\text{ odd}}{H_{n-1}(\xi)}$$

$$\frac{dH_n}{d\xi} \underset{n\text{ odd}}{=} \frac{d}{d\xi}\left[(-1)^{\frac{n-1}{2}} \frac{2n!}{(\frac{n-1}{2})!}\left(\xi - \frac{2(n-1)}{3!}\xi^3 + \frac{2^2(n-1)(n-3)}{5!}\xi^5 - \dots\right)\right]$$

$$= 2n\,(-1)^{\frac{n-1}{2}} \frac{(n-1)!}{(\frac{n-1}{2})!}\left(1 - \frac{2(n-1)}{2!}\xi^2 + \frac{2^2(n-1)(n-3)}{4!}\xi^4 - \dots\right)$$

$$= 2n\, \underset{(n-1)\text{ even}}{H_{n-1}(\xi)}$$

(b) $$\frac{d}{d\xi}F(\xi,s) = \sum_{n=0}^{\infty} \frac{dH_n}{d\xi}\frac{s^n}{n!} = 2\sum_{n=1}^{\infty} \frac{H_{n-1}}{(n-1)!}s^n = 2s\sum_{n=0}^{\infty}\frac{H_n}{n!}s^n = 2sF(\xi,s)$$

$$\Rightarrow F(\xi,s) = C' e^{2\xi s}$$

and $$F(0,s) = \sum_{n=0}^{\infty}\frac{H_n(0)}{n!}s^n = \underset{\substack{\text{only constant}\\ \text{term of even } n \text{ contribute}}}{\sum_{\substack{n=0\\ n\text{ even}}}^{\infty}} (-1)^{\frac{n}{2}}\frac{n!}{(\frac{n}{2})!}\frac{s^n}{n!} = \underset{n \mapsto 2n}{\sum_{n=0}^{\infty}}(-1)^n \frac{s^{2n}}{n!} = e^{-s^2} \equiv C'$$

$$\Rightarrow F(\xi,s) = e^{-s^2+2\xi s} = e^{\xi^2-(s-\xi)^2}$$

(c) $$H_n(\xi) = \frac{d^n}{ds^n}F(\xi,s)\Big|_{s=0} = e^{\xi^2}\frac{d^n}{ds^n}e^{-(s-\xi)^2}\Big|_{s=0} = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-(s-\xi)^2}\Big|_{s=0} = (-1)e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}$$

(d) Induction: $$\frac{d}{d\xi}\frac{2^n}{\sqrt{\pi}}\int_{\mathbb{R}}(\xi+is)^n e^{-s^2}ds = 2n\frac{2^{n-1}}{\sqrt{\pi}}\int_{\mathbb{R}}(\xi+is)^{n-1}e^{-s^2}ds$$

same as the $H_n(\xi)$, see (a); and

$$H_0(\xi) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} 1\cdot e^{-s^2}ds = 1 \quad ✓$$

Problem [2]

(a) $$I = \int_{\mathbb{R}} F(\xi, s) F(\xi, t)\, e^{-\xi^2 + 2\lambda\xi}\, d\xi =$$

$$\stackrel{(a)}{=} \int_{\mathbb{R}} e^{\xi^2 - (s-\xi)^2 + \xi^2 - (t-\xi)^2}\, e^{-\xi^2 + 2\lambda\xi}\, d\xi$$

$$= e^{\lambda^2 + 2(st + \lambda s + \lambda t)} \int_{\mathbb{R}} e^{-(\xi - s - t - \lambda)^2}\, d\xi = \sqrt{\pi}\, e^{\lambda^2 + 2(st + \lambda s + \lambda t)}$$

On the other hand $$I = \sum_{n,m,k=0}^{\infty} \int_{\mathbb{R}} H_n(\xi) \frac{s^n}{n!} H_m(\xi) \frac{t^m}{m!} e^{-\xi^2} \xi^k \frac{(2\lambda)^k}{k!}$$

$$\Rightarrow I_{nmk} = \int_{\mathbb{R}} H_n(\xi) H_m(\xi)\, e^{-\xi^2} \xi^k\, d\xi = \frac{1}{2^k} \frac{\partial^{n+m+k} I}{\partial s^n \partial t^m \partial \lambda^k}\Bigg|_{s,t,\lambda=0}$$

(b) $$\langle \psi_n | x \psi_{n'} \rangle = \int_{\mathbb{R}} \psi_n(x)\, x\, \psi_{n'}(x)\, dx = 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'!}} \sqrt{\frac{\hbar}{m\omega\pi}} \int H_n(\xi) H_{n'}(\xi) e^{-\xi^2} \xi\, d\xi$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\stackrel{(a)}{=} 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'!}} \sqrt{\frac{\hbar}{m\omega\pi}}\, I_{nn'1}$$

$$\underset{(a)}{=} 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'!}} \sqrt{\frac{\hbar}{m\omega}} \frac{1}{2} \frac{\partial^{n+n'}}{\partial s^n \partial t^{n'}}\Bigg|_{s,t=0} \underbrace{\left[2(s+t)\, e^{2st}\right]}_{\longrightarrow\, 2 \sum_{\ell=0}^{\infty} \left[\frac{s^{\ell+1}(2t)^\ell}{\ell!} + \frac{t^{\ell+1}(2s)^\ell}{\ell!}\right]}$$

$$= 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'!}} \sqrt{\frac{\hbar}{m\omega}} \sum_{\ell=0}^{\infty} \left[\delta_{n,\ell+1} \delta_{n',\ell}\, n!\, 2^{n'} + \delta_{n',\ell+1} \delta_{n,\ell}\, n'!\, 2^n\right]$$

$$= \sqrt{\frac{\hbar}{m\omega}} \left[\delta_{n,n'+1} \sqrt{n}\, 2^{-\frac{1}{2}} + \delta_{n',n+1} \sqrt{n'}\, 2^{-\frac{1}{2}}\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left[\delta_{n',n-1} \sqrt{n} + \delta_{n',n+1} \sqrt{n+1}\right]$$

(c) $$\langle \psi_n | x^2 \psi_{n'} \rangle \underset{\text{as in (b)}}{=} 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'!}} \frac{\hbar}{m\omega} \frac{1}{\sqrt{\pi}} I_{nn'2}$$

$$= 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!n'!}} \frac{\hbar}{m\omega} \frac{1}{4} \frac{\partial^{n+n'}}{\partial s^n \partial t^{n'}}\Bigg|_{s,t=0} \underbrace{\left[\left(2 + 4(s+t)^2\right) e^{2st}\right]}_{2 \sum_{\ell=0}^{\infty} \left[\frac{(2st)^\ell + 2(2st)^{\ell+1}}{\ell!} + \frac{2 s^{\ell+2}(2t)^\ell}{\ell!} + \frac{2t^{\ell+2}(2s)^{\ell}}{\ell!}\right]}$$

$$= 2^{-\frac{1}{2}(n+n')} \frac{1}{\sqrt{n!\,n'!}} \frac{\hbar}{im\omega} \frac{i}{2} \sum_{\ell=0}^{\infty} \Big[\delta_{n,\ell}\delta_{n',\ell} 2^{n} n! + \delta_{n,\ell+1}\delta_{n',\ell+1} 2^{n+1} n\, n! + \delta_{n,\ell+2}\delta_{n',\ell} 2^{n'+1} n! + \delta_{n',\ell+2}\delta_{n,\ell} 2^{n+1} n'! \Big]$$

$$= \frac{\hbar}{2m\omega}\left[\delta_{n,n'} + \delta_{n,n'} 2n + \delta_{n,n'+2}\sqrt{n(n-1)} + \delta_{n',n+2}\sqrt{n'(n'-1)} \right]$$

$$= \frac{\hbar}{2m\omega}\left[(2n+1)\delta_{n,n'} + \delta_{n',n-2}\sqrt{n(n-1)} + \delta_{n',n+2}\sqrt{(n+2)(n-1)} \right]$$

Problem [3]

[3] Let $\psi(x,0) = \sum_{n\in\mathbb{N}} c_n \psi_n(x)$ with ψ_n energy eigenstates

$$\psi(x,t) = \sum_{n\in\mathbb{N}} c_n \psi_n(x)\, e^{-\frac{i}{\hbar}((n+\frac{1}{2})\hbar\omega)t}$$

$$\langle x\rangle = \langle \psi(x,t) | x\ \psi(x,t)\rangle = \sum_{n,n'\in\mathbb{N}} c_n^* c_{n'} e^{-i(n'-n)\omega t} \langle \psi_n | x\, \psi_{n'}\rangle$$

$$\underset{[2](b)}{=} \sum_{n,n'} c_n^* c_{n'} e^{-i(n'-n)\omega t} \sqrt{\frac{\hbar}{2m\omega}} \left[\delta_{n',n-1}\sqrt{n} + \delta_{n',n+1}\sqrt{n+1} \right]$$

$$= \sum_{n\in\mathbb{N}} \left[c_n^* c_{n-1} e^{i\omega t}\sqrt{n} + c_{n-1}^* c_n \sqrt{n}\, e^{-i\omega t} \right] \sqrt{\frac{\hbar}{2m\omega}}$$

Write $c_i = |c_i| e^{i\phi_i}$ $\forall i \in \mathbb{N}$

$$\Rightarrow \langle x\rangle = \sum_{n\in\mathbb{N}} \sqrt{n}\, |c_n||c_{n-1}| \Big(\underbrace{e^{i(\omega t + \phi_{n-1} - \phi_n)} + e^{-i(\omega t + \phi_{n-1} - \phi_n)}}_{2\cos(\omega t + \phi_{n-1} - \phi_n)} \Big) \sqrt{\frac{\hbar}{2m\omega}} \qquad (*)$$

Obviously $\langle x\rangle(t=0) = \sqrt{\frac{2\hbar}{2m\omega}} \sum_{n\in\mathbb{N}} \sqrt{n}\, |c_n||c_{n-1}| \cos(\phi_{n-1} - \phi_n) \equiv \langle x\rangle_0$

Momentum op.: $p\psi_n(x) = -i\hbar\sqrt{\frac{m\omega}{\hbar}} \frac{d}{d\xi} C_n H_n(\xi) e^{-\xi^2/2} = -i\hbar\sqrt{\frac{m\omega}{\hbar}} C_n \left(2nH_{n-1}(\xi) - \xi H_n(\xi)\right) e^{-\xi^2/2}$

$$= i\hbar \frac{m\omega}{\hbar} x\psi_n(x) - i\hbar\sqrt{\frac{m\omega}{\hbar}}\sqrt{2n}\, \psi_{n-1}(x)$$

$$\Rightarrow \langle \psi_n | p\, \psi_{n'}\rangle = i\hbar\sqrt{\frac{m\omega}{\hbar}} \left[\delta_{n',n-1}\sqrt{\frac{n}{2}} - \delta_{n,n'-1}\sqrt{\frac{n}{2}} \right]$$

$$\langle p\rangle = \langle \psi(x,t) | p\, \psi(x,t)\rangle = i\hbar\sqrt{\frac{m\omega}{\hbar}} \sum_{n,n'} c_n^* c_{n'} e^{-i(n'-n)\omega t} \left[\delta_{n',n-1}\sqrt{\frac{n}{2}} - \delta_{n,n'-1}\sqrt{\frac{n'}{2}} \right]$$

$$= i\hbar\sqrt{\frac{m\omega}{\hbar}} \sum_{n\in\mathbb{N}} \left[c_n^* c_{n-1} e^{i\omega t}\sqrt{\frac{n}{2}} - c_{n-1}^* c_n e^{-i\omega t}\sqrt{\frac{n}{2}} \right]$$

$$= -\sqrt{2\hbar m\omega} \sum_{n\in\mathbb{N}} \sqrt{n}\, |c_n||c_{n-1}| \sin(\omega t + \phi_{n-1} - \phi_n)$$

$$\Rightarrow \langle p\rangle(0) = -\sqrt{2\hbar m\omega} \sum_{n\in\mathbb{N}} \sqrt{n}\,|c_n||c_{n-1}| \sin(\phi_{n-1}-\phi_n) \equiv \langle p\rangle_0$$

and $\langle p\rangle = m \frac{d}{dt}\langle x\rangle$ from (*)

also from (*): $\langle x\rangle = \langle x\rangle_0 \cos\omega t + \frac{\langle p\rangle_0}{m\omega} \sin\omega t$

(b) $p^2 = 2m(H-V) = 2mH - m^2\omega^2 x^2$ as operators

$$\Rightarrow \langle\psi_n|p^2\psi_{n'}\rangle = 2m\langle\psi_n|H\psi_{n'}\rangle - m^2\omega^2\langle\psi_n|x^2\psi_{n'}\rangle$$

$\Rightarrow \langle p^2\rangle = \langle\psi_n|p^2\psi_n\rangle = \underbrace{2m\hbar\omega(n+\tfrac{1}{2})}_{2m\langle H\rangle} - \underbrace{\hbar m\omega\tfrac{1}{2}(2n+1)}_{2m\langle V\rangle} = m\hbar\omega(n+\tfrac{1}{2})$ for diagonal case $n=n'$

In particular $\langle T\rangle = \langle V\rangle$

Virial theorem: $\langle T\rangle = \frac{1}{2}\langle x\,\frac{dV}{dx}\rangle = \langle V\rangle$ for harm. osc. ✓

For case $n\neq n'$: $\langle\psi_n|p^2\psi_{n'}\rangle = -m^2\omega^2\langle\psi_n|x^2\psi_{n'}\rangle$, cf. [2](c)