PHYS 606 – Spring 2017 – Homework V – Solution

Problem [1]

(a) Simultaneous momentum eigenstates: separation $\psi(\vec{r}) = X(x)\,Y(y)\,Z(z)$

$$-i\hbar \frac{\partial}{\partial x}\psi(x,y,z) = \underbrace{p_x}_{\text{eigenvalue}}\,\psi(x,y,z) \quad \Rightarrow \quad X(x) = \text{const} * e^{\frac{i}{\hbar}p_x x}, \quad p_x \in \mathbb{R}$$

(p_x hermitian op.)

Same for y, z-directions

$$\Rightarrow \psi(x,y,z) = \text{const.} * e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$$ eigenfct. for p_x, p_y, p_z simultaneously

with $(p_x, p_y, p_z) \in \mathbb{R}^3$

Instead of the separation ansatz you can also use our results for free particles

Impose boundary conditions: for periodic boundary cond. on box of size L the fct. $\psi(\vec{r})$ needs to be periodic w/ period L:

$$\psi(x,y,z) = \psi(x+n_1L,\, y+n_2L,\, z+n_3L) \quad \text{with } (n_1,n_2,n_3) \in \mathbb{Z}^3$$

$$\Rightarrow \frac{i}{\hbar}p_i L = 2\pi i n_i \Rightarrow p_i = \frac{2\pi\hbar}{L}n_i = \frac{h}{L}n_i$$

$\Rightarrow$ eigenvalues that obey B.c. are $\vec{p}_N = \frac{h}{L}\underbrace{(n_1, n_2, n_3)}_{=:N \in \mathbb{Z}^3}$

with eigenfcts. $\psi_N(\vec{r}) = \underbrace{L^{-3/2}}_{\text{normalization from } \int_{\text{Box}} |\psi_N|^2 d^3r = 1} e^{\frac{i}{\hbar}\vec{p}_N\cdot\vec{r}}$

$H\psi_N(\vec{r}) = \frac{p_N^2}{2m}\psi_N(\vec{r}) \Rightarrow \psi_N$ are eigenstate of H with eigenvalues

$$E_N = \frac{h^2}{2mL^2}(n_1^2+n_2^2+n_3^2) \geq 0$$

For $L \to \infty$ the lattice of the $\vec{p}_N = \frac{h}{L}N$ has lattice spacing $\frac{h}{L} \to 0$

i.e. it will cover all of $\mathbb{R}^3$ with eigenfcts. $\sim e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}}$ (as in I.II.4, case (B))

(b) Number of states in a $\Delta n_x \Delta n_y \Delta n_z$ cube of the lattice is $\Delta n_x \Delta n_y \Delta n_z = \Delta N$.

Its momentum space volume is $\Delta p_x \Delta p_y \Delta p_z = \frac{h^3}{L^3} \Delta n_x \Delta n_y \Delta n_z$

$\Rightarrow \rho = \frac{\Delta N}{L^3 \Delta p_x \Delta p_y \Delta p_z} = \frac{1}{h^3}$, i.e. we have one eigenstate per elementary phase space volume h^3

(c) An energy interval ΔE cuts out a spherical shell in the $\mathbb{Z}^3$ lattice of $\overset{\text{momentum}}{\text{eigenvalues}}$.

The number of eigenvalues in that shell grows quadratically with its radius.

For large E (large radius) we should be able to approximate the sum over points in the shell by an integral over the shell

Number of eigenstates $\Delta N \approx \underset{\substack{\uparrow \\ \text{phase space} \\ \text{density}}}{\rho} L^3 \underbrace{4\pi p^2 \Delta p}_{\text{shell volume in } \vec{p}\text{-space}}$ for large p but $\Delta p \ll p$

$p^2 = 2mE \Rightarrow \Delta p = \frac{m \Delta E}{p}$

$\Rightarrow \sigma = \frac{\Delta N}{\Delta E} = \frac{L^3}{h^3} 4\pi \sqrt{2mE}\, m = \frac{m^{3/2} L^3}{\sqrt{2}\, \pi^2 \hbar^3} \sqrt{E}$

Problem [2]

(a) Eigenvalues and eigenfcts. for mom. operator p are usually $K \in \mathbb{R}$ and $\sim e^{\frac{i}{\hbar}Kx}$. Since p and U_a commute try the same eigenfcts:

$$U_a e^{\frac{i}{\hbar}Kx} = \sum_{j=0}^{\infty} \left(\frac{i}{\hbar} p a\right)^j e^{\frac{i}{\hbar}Kx} = e^{-\frac{i}{\hbar}Ka} \, e^{\frac{i}{\hbar}Kx}$$

$\Rightarrow$ eigenvalues are $e^{\frac{i}{\hbar}Ka}$ with eigenfct. $\sim e^{\frac{i}{\hbar}Kx}$

But eigenvalues $e^{-\frac{i}{\hbar}Ka}, e^{-\frac{i}{\hbar}K'a}$ with $K - K' = \frac{2\pi\hbar}{a}$ are actually the same

$\Rightarrow$ The set $e^{-\frac{i}{\hbar}Ka}$ with $-\frac{h}{2a} < K < \frac{h}{2a}$ is a unique set of eigenvalues, each with an infinite but countable degeneracy. As a set it covers the unit circle in $\mathbb{C}$.

The fcts. $e^{\frac{i}{\hbar}(K + \frac{2\pi\hbar n}{a})x}$ for $n \in \mathbb{Z}$ have all the same eigenvalue $e^{-\frac{i}{\hbar}Ka}$, and they are mutually orthogonal.

(b) Fourier's Theorem: the fcts. $e^{i 2\pi n \frac{x}{a}}$ $n \in \mathbb{Z}$ are periodic with period a. They are a complete basis spanning the space of square-integrable periodic fcts with period a.

Thus: arbitrary fct. in eigenspace of eigenvalue $e^{-\frac{i}{\hbar}Ka}$ is a linear combination

$$\sum_{n \in \mathbb{Z}} c_n e^{\frac{i}{\hbar}(K + \frac{2\pi\hbar n}{a})x} = e^{\frac{i}{\hbar}Kx} \underbrace{\sum_{n \in \mathbb{Z}} c_n e^{i 2\pi n \frac{x}{a}}}_{\text{Fourier series}} = e^{\frac{i}{\hbar}Kx} \underbrace{u(x)}_{a\text{-periodic fct.}}$$

Problem [3]

(a) $$\mathcal{D}_{w_i} = e^{\frac{i}{\hbar}(mr_i - p_i t)w_i} = e^{\frac{i}{\hbar}(mr_i - p_i t)w_i + \overbrace{\left[\frac{i}{\hbar}mr_iw_i, -\frac{i}{\hbar}p_iw_it\right]/2}^{\frac{i}{2\hbar}mw_i^2t}} e^{-\frac{i}{2\hbar}mw_i^2t}$$

exponent just a number, so can be pulled out

just numbers

$$\underset{BCH}{=} e^{\frac{i}{\hbar}mr_iw_i} e^{-\frac{i}{\hbar}p_iw_it} e^{-\frac{i}{\hbar}\frac{mw_i^2}{2}t}$$

commutes here

$$\Rightarrow \mathcal{D}_{w_i}\psi(\vec{r},t) = e^{\frac{i}{\hbar}(mw_ir_i - \frac{mw_i^2}{2}t)} \underbrace{e^{-\frac{i}{\hbar}p_iw_it}}_{\substack{\text{translation}\\ \text{operator by}\\ -\vec{w}t}} \psi(\vec{r},t) = e^{\frac{i}{\hbar}(mw_ir_i - \frac{mw_i^2}{2}t)}\psi(\vec{r}-\vec{w}t,t)$$

(b) $$[K_i, K_j] = [mr_i - p_it, mr_j - p_jt] = 0$$

$$[K_i, p_j] = [mr_i, p_j] = i\hbar m\,\delta_{ij}$$

$$[K_i, H] = [mr_i, T] \underset{\text{I.8.2}}{=} i\hbar\, p_i$$

(c) $$e^{\frac{i}{\hbar}\vec{K}\vec{w}_2} e^{\frac{i}{\hbar}\vec{K}\cdot\vec{w}_1} \underset{(b)}{=} e^{\frac{i}{\hbar}\vec{K}(\vec{w}_2+\vec{w}_1)}$$

$$e^{\frac{i}{\hbar}\vec{K}\cdot\vec{w}_1} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{a}} = e^{\frac{i}{\hbar}(\vec{K}\cdot\vec{w}_1 + \vec{p}\cdot\vec{a} + \frac{i}{\hbar}w_{1j}\overbrace{[K_j, P_\ell]}^{i\hbar m\delta_{j\ell}}a_\ell/2)}$$

$$= \underbrace{e^{-\frac{i}{\hbar}\frac{m}{2}\vec{w}\cdot\vec{a}}}_{} e^{\frac{i}{\hbar}(\vec{K}\cdot\vec{w}_1 + \vec{p}\cdot\vec{a})}$$

appearance of this phase factor means this is a projective representation.