PHYS 606 – Spring 2017 – Homework IV – Solution

Problem [1]

Let's use the trick suggested by Merzbacher (other ways, e.g. calculating $\log(e^F e^G)$ using the series for the logarithm can be found in the literature.)

For $t \in \mathbb{R}$

Consider $\frac{d}{dt} e^{tF} e^{tG} = e^{tF} F e^{tG} + \underbrace{e^{tF} G}_{} e^{tG} = \left(F + G + \frac{1}{2}[F,G]\right) e^{tF} e^{tG}$

usual product rule or explicitly via power series

$= \left(G + \frac{1}{2!}[F,G]\right) e^{tF}$

According to (a)! Higher order commutators vanish

This is a diff. equation $\frac{d\varphi}{dt} = \text{const} * \varphi$

with $\varphi = e^{tF} e^{tG}$. The solution must be of the form $e^{t * \text{const}}$.

$\Rightarrow e^{tF} e^{tG} = e^{t(F+G+\frac{1}{2}[F,G])}$. Now set $t = 1$.

Problem [2]

$$(a)\ [\vec{r}\cdot\vec{p}, H] = p_i \left[\vec{r}\cdot\vec{p}, \frac{p_i}{2m}\right] + \left[\vec{r}\cdot\vec{p}, \frac{p_i}{2m}\right] p_i + r_i [p_i, V] + \overbrace{[r_i, V]}^{=0} p_i$$

$$= p_i r_j \underbrace{\left[p_j, \frac{p_i}{2m}\right]}_{=0} + p_i \underbrace{\left[r_j, \frac{p_i}{2m}\right]}_{\frac{i\hbar}{2m}\delta_{ij}} p_j + \underbrace{\left[r_j, \frac{p_i}{2m}\right]}_{\frac{i\hbar}{2m}\delta_{ij}} p_j p_i + \vec{r}\cdot(-i\hbar\nabla V)$$

$$= i\hbar \frac{p^2}{m} - i\hbar\, \vec{r}\cdot\nabla V = i\hbar\left(2T - \vec{r}\,\nabla V\right)$$

$[\vec{p}\cdot\vec{r}, H] =$ ... completely analogous ... $= i\hbar(2T - \vec{r}\cdot\nabla V)$

(b) $\left[(\vec{r}\cdot\vec{p})(\vec{r}\cdot\vec{p}), T\right] = (\vec{r}\cdot\vec{p})\underbrace{\left[\vec{r}\cdot\vec{p}, T\right]}_{\stackrel{(a)}{2i\hbar T}} + \underbrace{\left[\vec{r}\cdot\vec{p}, T\right]}_{\stackrel{(a)}{2i\hbar T}}\vec{r}\cdot\vec{p} = 2i\hbar\,(\vec{r}\cdot\vec{p}\,T + T\,\vec{r}\cdot\vec{p})$

$= 2i\hbar\,\{\vec{r}\cdot\vec{p}, T\}$

this sometimes denotes the anti commutator

$\left[(\vec{r}\cdot\vec{p})(\vec{p}\cdot\vec{r}), T\right] = 2i\hbar\,(\vec{r}\cdot\vec{p}\,T + T\,\vec{p}\cdot\vec{r})$

To see that both results are not the same apply these operators to test fcts. Here $\psi = e^{\frac{i}{\hbar}(\vec{r}\cdot\vec{p} - Et)}$ plane wave.

$T\psi = -\hbar^2 \frac{\Delta\psi}{2m} = \frac{p^2}{2m}\psi$; $(\vec{r}\cdot\vec{p})\psi = -i\hbar\,\vec{r}\cdot\nabla\psi = (\vec{r}\cdot\vec{p})\psi$

$\vec{p}$ operator! $\quad$ $\vec{p}$ number, an eigenvalue of the operator $\vec{p}$!

$(\vec{p}\cdot\vec{r})\psi = \vec{p}\cdot\vec{r}\,\psi - i\hbar\underbrace{(\nabla\vec{r})}_{3}\psi$

operator $\quad$ number

one $\vec{p}$ in T applied to $\vec{r}\cdot\vec{p}$

$\Rightarrow \left[(\vec{r}\cdot\vec{p})^2, T\right]\psi = 2i\hbar\left(\vec{r}\cdot\vec{p}\,\frac{p^2}{2m} + \frac{p^2}{2m}\,\vec{r}\cdot\vec{p} - 2i\hbar\frac{p^2}{2m}\right)\psi = \left(\frac{2i\hbar}{m}\vec{r}\cdot\vec{p}\,p^2 + \frac{2\hbar^2 p^2}{m}\right)\psi$

all numbers

$\left[(\vec{r}\cdot\vec{p})(\vec{p}\cdot\vec{r}), T\right]\psi = 2i\hbar\left(\vec{r}\cdot\vec{p}\,\frac{p^2}{2m} + \frac{p^2}{2m}\left(\vec{p}\cdot\vec{r} - 3i\hbar\right)\right)\psi = \left(\frac{2i\hbar}{m}(\vec{r}\cdot\vec{p})p^2 + \frac{5\hbar^2}{m}p^2\right)\psi$

$-2i\hbar\frac{p^2}{2m}$

The two operators defined by the commutators are not the same.

c) $\{\vec{r}\cdot\vec{p}, H\} = \sum_{k=1}^{3}\left(p_k \underbrace{\frac{\partial H}{\partial p_k}}_{p_k/m} - r_k \underbrace{\frac{\partial H}{\partial r_k}}_{\partial V/\partial r_k}\right) = \frac{p^2}{m} - \vec{r}\cdot\nabla V = 2T - \vec{r}\cdot\nabla V$

$\{r_i, p_j\} = \sum_{k=1}^{3}\left(\delta_{ik}\delta_{jk} - 0\right) = \delta_{ij}$

So it seems $\underbrace{[A, B]}_{\text{operators}} = i\hbar\,\underbrace{\{A, B\}}_{\text{cl. quantities}}$

(d) $\{(\vec{r}\cdot\vec{p})^2, T\} = \sum_{k=1}^{3}\left(2(\vec{r}\cdot\vec{p})\,p_k\,\frac{p_k}{m} - 0\right) = 4\,\vec{r}\cdot\vec{p}\,T$

Caveat: obviously the order matters: $\frac{1}{i\hbar}\left[(\vec{r}\cdot\vec{p})^2, H\right] = 4\,\frac{1}{2}(\vec{r}\cdot\vec{p}\,T + T\,\vec{r}\cdot\vec{p}) \neq 4\,\vec{r}\cdot\vec{p}\,T$ for operators

[3] (a) Hamilton-Jacobi: $\frac{\partial S}{\partial t} + \underbrace{\frac{1}{2m}\left(\nabla S - q\vec{A}\right)^2}_{H \text{ with } \vec{p} = \nabla S} + q\phi = 0$

Continuity equ.: $\frac{\partial \varrho}{\partial t} + (\nabla \vec{v})\,\varrho + \vec{v}\cdot\nabla\varrho = 0 \Rightarrow \frac{\partial \varrho}{\partial t} + \frac{\varrho}{m}\nabla(\vec{p} - q\vec{A}) + \frac{1}{m}\nabla\varrho\cdot(\vec{p} - q\vec{A}) = 0$

(I.5.3)

(b) Ansatz $\psi(\vec{r},t) = C e^{\frac{i}{\hbar}S}$ with C, S real into

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2\psi + q\phi\psi$$

$$\Rightarrow \left(i\hbar\frac{1}{C}\frac{\partial C}{\partial t} - \frac{\partial S}{\partial t}\right)\psi = \frac{1}{2m}\Big(-\hbar^2\frac{\Delta C}{C} + (\nabla S)^2 - i\hbar\Delta S - i\hbar\frac{2}{C}\nabla C\cdot\nabla S + q^2A^2$$
$$+ i\hbar\frac{2}{C}q\vec{A}\cdot\nabla C - 2q\vec{A}\cdot\nabla S + i\hbar q\nabla\vec{A}\Big)\psi + q\phi\psi$$

Drop ψ from eq. and separate imaginary and real part:

Re: $0 = \frac{\partial S}{\partial t} + \frac{1}{2m}\Big((\nabla S)^2 - \underbrace{\hbar^2\frac{\Delta C}{C}}_{0 \text{ for } \hbar\to 0} - 2\nabla S\cdot(q\vec{A}) + (q\vec{A})^2\Big) + q\phi$

$$\Rightarrow 0 = \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\nabla S - q\vec{A}\right)^2 + q\phi \qquad \text{Hamilton-Jacobi!}$$

Im: $\underbrace{2C\frac{\partial C}{\partial t}}_{\frac{\partial \varrho}{\partial t}} = \frac{1}{2m}\Big(-2\underbrace{C^2}_{\varrho}\underbrace{\Delta S}_{\nabla\vec{p}} - \underbrace{4C\nabla C}_{2\nabla\varrho}\cdot\underbrace{\nabla S}_{\vec{p}} + \underbrace{4C(\nabla C)}_{2\nabla\varrho}\cdot(q\vec{A}) + 2\underbrace{C^2}_{\varrho}q\nabla\vec{A}\Big)$

where $\varrho = C^2 = |\psi|^2$; $\Rightarrow \frac{\partial \varrho}{\partial t} + \frac{\varrho}{m}\nabla(\vec{p} - q\vec{A}) + \frac{1}{m}\nabla\varrho\cdot(\vec{p} - q\vec{A}) = 0$

continuity equation.

[4] Define $\vec{A}' = \vec{A} + \nabla f$; $\phi' = \phi - \frac{\partial f}{\partial t}$; $\psi' = \psi e^{\frac{i}{\hbar} q f}$

Proof of invariance in two parts:

i) $i\hbar \frac{\partial \psi'}{\partial t} - q\phi'\psi' = \left(i\hbar\frac{\partial \psi}{\partial t}\right) e^{\frac{i}{\hbar} q f} + \cancel{\left(-q\frac{\partial f}{\partial t}\right)\psi e^{\frac{i}{\hbar} q f}} - q\phi\psi e^{\frac{i}{\hbar} q f} + \cancel{q\frac{\partial f}{\partial t}\psi e^{\frac{i}{\hbar} q f}}$

$= \left(i\hbar\frac{\partial \psi}{\partial t} - q\phi\psi\right) e^{\frac{i}{\hbar} q f}$

ii) $\left(-i\hbar\nabla - q\vec{A}'\right)^2 \psi' = \left(-i\hbar\nabla - q\vec{A}\right)^2 \psi e^{\frac{i}{\hbar} q f} + q^2(\nabla f)^2 \psi e^{\frac{i}{\hbar} q f} + \left(-i\hbar\nabla - q\vec{A}\right)(-q\nabla f)\psi e^{\frac{i}{\hbar} q f}$

$+ (-q\nabla f)\left(-i\hbar\nabla - q\vec{A}\right)\psi e^{\frac{i}{\hbar} q f}$

$= \left[\left(-i\hbar\nabla - q\vec{A}\right)^2\psi\right] e^{\frac{i}{\hbar} q f} + 2\left[\left(-i\hbar\nabla - q\vec{A}\right)\psi\right]\underbrace{(-i\hbar\nabla) e^{\frac{i}{\hbar} q f}}_{q\nabla f e^{\frac{i}{\hbar} q f}} + \left[\underbrace{(-i\hbar\nabla)^2 e^{\frac{i}{\hbar} q f}}_{q^2(\nabla f)^2 e^{\frac{i}{\hbar} q f} - i\hbar q \Delta f e^{\frac{i}{\hbar} q f}}\right]\psi$

$+ \cancel{q^2(\nabla f)^2\psi e^{\frac{i}{\hbar} q f}} + \cancel{i\hbar q \Delta f \psi e^{\frac{i}{\hbar} q f}} - \cancel{2q^2(\nabla f)^2 \psi e^{\frac{i}{\hbar} q f}} - \cancel{\left[2q\nabla f \cdot \left(-i\hbar\nabla - q\vec{A}\right)\psi\right] e^{\frac{i}{\hbar} q f}}$

$= \left[\left(-i\hbar\nabla - q\vec{A}\right)^2\psi\right] e^{\frac{i}{\hbar} q f}$

$\Rightarrow$ the transformed S.E. $i\hbar\frac{\partial \psi'}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}'\right)^2\psi' + q\phi'\psi'$

is identical to the original equation times an overall phase factor which can be dropped.