PHYS 606 – Spring 2017 – Homework III – Solution

Problem [1]

(a) Induction: eqs (2) + (3) obviously true for $n=1$; suppose they are true for $n-1$

then

$$\left\{\begin{array}{l} [F, G^n] = G^{n-1}[F,G] + [F, G^{n-1}]G = G^{n-1}[F,G] + (n-2)G^{n-2}\underbrace{[F,G]G}_{G \text{ commutes w/ } [F,G]} \\ [F^n, G] \underset{\uparrow}{=} F^{n-1}[F,G] + [F^{n-1},G]F = F^{n-1}[F,G] + (n-2)F^{n-2}[F,G]F \end{array}\right\}$$

Product rule

$$= \begin{cases} (n-1)\, G^{n-1}\, [F,G] \\ (n-1)\, F^{n-1}\, [F,G] \end{cases}$$

Direct calculation is also possible.

(b) $$[F,[G,H]] = [F, GH] - [F, HG] \underset{|}{=} G[F,H] + [F,G]H - H[F,G] - [F,H]G$$

Product rule

$$= [G,[F,H]] - [H,[F,G]] = -[H,[F,G]] - [G,[H,F]]$$

Problem [2]

$$[F_{\vec r}(\vec r), G_{\vec r}(\vec p)]\, f(\vec r) = (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} [F(\vec r), G(-i\hbar\nabla_r)]\, \hat f(\vec p)\, e^{\frac{i}{\hbar}\vec r\cdot\vec p}\, d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \Big(F(\vec r)\, \hat f(\vec p)\, \underbrace{G(\vec p)\, e}_{\substack{\text{power series of}\\ \text{G has acted on}\\ \text{phase factor}}}{}^{\frac{i}{\hbar}\vec r\cdot\vec p} - G(-i\hbar\nabla_r)\, \hat f(\vec p)\, \underbrace{F(-i\hbar\nabla_p)\, e}_{\substack{\text{power series}\\ \text{of F acting}\\ \text{on phase factor}}}{}^{\frac{i}{\hbar}\vec r\cdot\vec p} \Big)\, d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \Big(G(\vec p)\, \hat f(\vec p)\, \underline{F(-i\hbar\nabla_p)\, e}{}^{\frac{i}{\hbar}\vec r\cdot\vec p} - \hat f(\vec p)\, F(-i\hbar\nabla_p)\, \underline{G(\vec p)}\, e^{\frac{i}{\hbar}\vec r\cdot\vec p} \Big)\, d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} \Big(\underbrace{\big(F(+i\hbar\nabla_p)\, G(\vec p)\, \hat f(\vec p)\big)}_{\substack{\text{multiple partial integrations}\\ \text{according to the power series}\\ \text{of F, boundary terms vanish}\\ \text{for sufficiently fast falling}\\ \text{functions.}}} e^{\frac{i}{\hbar}\vec r\cdot\vec p} - G(\vec p)\, \underbrace{\big(F(+i\hbar\nabla_p)\, \hat f(\vec p)\big)}_{\text{partial integrations}}\, e^{\frac{i}{\hbar}\vec r\cdot\vec p} \Big)\, d^3p$$

$$= (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} [F_{\vec p}(\vec r), G_{\vec p}(\vec p)]\, \hat f(\vec p)\, e^{\frac{i}{\hbar}\vec r\cdot\vec p}\, d^3p$$

[3] $\frac{d}{dt}\langle p\rangle = \underset{\text{(Ehrenfest)}}{\langle F\rangle} = 0$ (free particle) $\Rightarrow \langle p\rangle(t) = \underbrace{\langle p\rangle(0)}_{=:\, p_0 \text{ short notation}} = \text{const.}$

$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m} = \frac{p_0}{m}$

$\Rightarrow \langle x\rangle(t) = \frac{p_0}{m}t + x_0 \qquad x_0 = \langle x\rangle(0)$

$\frac{d}{dt}\langle p^2\rangle = \frac{1}{i\hbar}\langle[p, \underset{\frac{p^2}{2m}}{T}]\rangle = 0 \qquad \Rightarrow \langle p^2\rangle(t) = \langle p^2\rangle(0) = \text{const.}$

$\Rightarrow (\Delta p)^2(t) = \langle (p-\langle p\rangle)^2\rangle = \langle p^2\rangle(t) - (\langle p\rangle(t))^2 = \langle p^2\rangle(0) - p_0^2 = (\Delta p)^2(0) = \text{const.}$

$\frac{d}{dt}\langle x^2\rangle = \frac{1}{i\hbar}\langle[x^2, \frac{p^2}{2m}]\rangle = \frac{1}{i\hbar}\langle p[x^2, \frac{p}{2m}] + [x^2, \frac{p}{2m}]p\rangle$

$= \frac{1}{i\hbar}\langle px[x,p] + p[x,p]x + x[x,p]p + [x,p]xp\rangle \frac{1}{2m}$

$= \frac{1}{m}\langle px + xp\rangle$

$\frac{d}{dt}\langle px\rangle = \frac{1}{i\hbar}\langle[px, T]\rangle \overset{[2](a)}{=} 2T = \text{const.} = \frac{d}{dt}\langle xp\rangle$

$\Rightarrow \langle px + xp\rangle(t) = \frac{2\langle p^2\rangle(0)}{m}t + \langle px + xp\rangle(0)$

$\Rightarrow \langle x^2\rangle = \frac{\langle p^2\rangle(0)}{m^2}t^2 + \frac{1}{m}\langle px + xp\rangle(0)\, t + \langle x^2\rangle(0)$

$\Rightarrow (\Delta x)^2(t) = \langle x^2\rangle(t) - (\langle x\rangle(t))^2 = \frac{\langle p^2\rangle(0) - p_0^2}{m^2}t^2 + \frac{t}{m}\left(\langle px + xp\rangle(0) - 2x_0p_0\right)$
$\qquad + \langle x^2\rangle(0) - x_0^2$

$= \frac{(\Delta p)^2(t)}{m^2}t^2 + \frac{2}{m}\left(\frac{1}{2}\langle px + xp\rangle(0) - \langle p\rangle(0)\langle x\rangle(0)\right)t + (\Delta x)^2(0)$

[4] (a) Gauss: $(\Delta x)^2(0) = \sigma^2$, $(\Delta p)^2(0) = \frac{\hbar^2}{4\sigma^2}$ (HW I, [3])

$\langle x\rangle(0) = x_0$, $\langle p\rangle(0) = \hbar k_0 = p_0$

$$\langle xp\rangle(0) = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{-\frac{i}{\hbar}p_0x}\, x\left(-i\hbar\frac{d}{dx}\right) e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{+\frac{i}{\hbar}p_0x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}\, x\left(p_0 + i\hbar\frac{x-x_0}{2\sigma^2}\right) dx$$

$$\underset{u=x-x_0}{=} \frac{1}{\sqrt{2\pi}\sigma}\int e^{-\frac{u^2}{2\sigma^2}}\left(\cancel{up_0} + i\hbar\frac{u^2}{2\sigma^2} + x_0 p_0 + i\hbar x_0 \cancel{\frac{u}{2\sigma^2}}\right) d$$

$$= x_0 p_0 + i\hbar\frac{\sigma^2}{2\sigma^2} = x_0 p_0 + \frac{1}{2}i\hbar$$

$\langle px\rangle(0) =$ same as above + term with $(-i\hbar\frac{d}{dx})$ acting on x

$$= x_0 p_0 + \frac{1}{2}i\hbar + \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} e^{-\frac{u^2}{2\sigma^2}}(-i\hbar) = x_0 p_0 - \frac{1}{2}i\hbar$$

$$\Rightarrow \frac{1}{2}\langle px + xp\rangle(0) = x_0 p_0$$

$$\Rightarrow (\Delta x)^2(t) = \frac{\hbar^2}{4\sigma^2 m^2}t^2 + \sigma^2 \quad \text{and } (\Delta p)^2(t) = \frac{\hbar^2}{4\sigma^2}$$

(b) $$(\Delta p)^2(t) \overset{HW\,II,\,[1](a)}{=} \frac{1}{\sqrt{2\pi}\hat{\sigma}}\int e^{-\frac{(p-p_0)^2}{4\hat{\sigma}^2}}\left|e^{-\frac{i}{\hbar}(p-p_0)x_0}e^{-i\omega(p)t}\right|^2(p-p_0)^2\,dp \quad \left(\hat{\sigma} = \frac{\hbar}{2\sigma}\right)$$

$$= (\Delta p)^2(0) = \hat{\sigma}^2 = \frac{\hbar^2}{4\sigma^2}$$

HW II, [1](a): $$|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}}\frac{\sigma}{\sqrt{\sigma^4 + \frac{\hbar^2 t^2}{4m^2}}}\, e^{-\frac{(x-x_0-v_{gr}t)^2\sigma^2}{2(\sigma^4+\frac{\hbar^2t^2}{4m^2})}} \qquad v_{gr} = \frac{p_0}{m}$$

$$\langle x\rangle(t) \underset{x-x_0-v_{gr}t=:u}{=} \frac{1}{\sqrt{2\pi}}\frac{\sigma}{\sqrt{\sigma^4+\frac{\hbar^2t^2}{4m^2}}}\int_{\mathbb{R}} du\,\underbrace{(x_0+v_{gr}t+\cancel{u})}_{=x}\, e^{-\frac{u^2\sigma^2}{2(\sigma^4+\frac{\hbar^2t^2}{4m^2})}} = x_0 + v_{gr}t$$

$$\langle x^2\rangle(t) = \frac{1}{\sqrt{2\pi}}\frac{\sigma}{\sqrt{\sigma^4+\frac{\hbar^2t^2}{4m^2}}}\int_{\mathbb{R}} du\,\left((x_0+v_{gr}t)^2 + \cancel{2u(x_0+v_{gr}t)} + u^2\right)e^{-\frac{u^2\sigma^2}{2(\sigma^4+\frac{\hbar^2t^2}{4m^2})}}$$

$$= (x_0+v_{gr}t)^2 + \frac{\sigma^4+\frac{\hbar^2t^2}{4m^2}}{\sigma^2}$$

$$\Rightarrow \langle (x-\langle x\rangle)^2\rangle = \langle x^2\rangle(t) - (\langle x\rangle(t))^2 = \sigma^2 + \frac{\hbar^2}{4\sigma^2m^2}t^2$$