PHYS 606 – Spring 2017 – Homework X – Solution

Problem [1]

a) $[L_i, r_j] = \varepsilon_{ik\ell}\,[r_k p_\ell, r_j] = \varepsilon_{ik\ell}\left(r_k \underbrace{[p_\ell, r_j]}_{-i\hbar\delta_{j\ell}} + \underbrace{[r_k, r_j]}_{=0} p_\ell\right) = i\hbar\,\varepsilon_{ijk}\, r_k$

$[L_i, p_j] = \varepsilon_{ik\ell}\left(r_k \underbrace{[p_\ell, p_j]}_{=0} + \underbrace{[r_k, p_j]}_{i\hbar\delta_{kj}} p_\ell\right) = i\hbar\,\varepsilon_{ij\ell}\, p_\ell$

$[L_i, K_j] = [L_i, m r_j - p_j t] = i\hbar\,\varepsilon_{ijk}\left(m r_k - p_k t\right) = i\hbar\,\varepsilon_{ijk} K_k$

b) $[L_\pm, L_z] = \underbrace{[L_x, L_z]}_{-i\hbar L_y} \pm i \underbrace{[L_y, L_z]}_{i\hbar L_x} = \mp\hbar(L_x \pm i L_y) = \mp\hbar L_\pm$

$[L_+, L_-] = i[L_y, L_x] - i[L_x, L_y] = 2i(-i\hbar L_z) = 2\hbar L_z$

$L^2 - L_z^2 = L_x^2 + L_y^2 = (L_x \pm i L_y)(L_x \mp i L_y) - L_x(\mp i L_y) - (\pm i L_y) L_x$

$= L_\pm L_\mp \pm i \underbrace{[L_x, L_y]}_{i\hbar L_z} = L_\pm L_\mp \mp \hbar L_z$

Problem [2]

(a) $\langle n'|a|n\rangle = C_n \langle n'|n-1\rangle = C_n\,\delta_{n', n-1} \qquad \forall\, n, n' \in \mathbb{N}$

$\langle n'|a^\dagger|n\rangle = D_n\,\delta_{n', n+1}$

I.e. in explicit matrix form

$$a = \begin{pmatrix} 0 & C_1 & 0 & 0 & \cdots \\ 0 & 0 & C_2 & 0 & \cdots \\ 0 & 0 & 0 & C_3 & \cdots \\ & & \cdots & & \end{pmatrix} \qquad a^\dagger = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ D_0 & 0 & 0 & \cdots \\ 0 & D_1 & 0 & \cdots \\ 0 & 0 & D_2 & \\ & & \cdots & \end{pmatrix}$$

(b) $|C_n|^2 = |\langle n-1|a|n\rangle|^2 = \langle n|a^\dagger|n-1\rangle\langle n-1|a|n\rangle = \sum_{n' \in \mathbb{N}} \langle n|a^\dagger|n'\rangle\langle n'|a|n\rangle$

all matrix elements for $n' \neq n-1$ vanish

$= \langle n|a^\dagger a|n\rangle = n$

$\Rightarrow C_n = \sqrt{n}$ (choose phase to be zero)

Similarly $|D_n|^2 = |\langle n+1|a^+|n\rangle|^2 = \sum_{n'\in\mathbb{N}} \langle n|a|n'\rangle\langle n'|a^+|n\rangle$

$= \langle n|aa^+|n\rangle = \langle n|a^+a+\mathbb{1}|n\rangle = n+1$

$\Rightarrow D_n = \sqrt{n+1}$

$|n\rangle = N_n (a^+)^n |0\rangle = N_n \sqrt{n}\cdot\sqrt{n-1}\cdot\ldots|0\rangle = N_n \sqrt{n!}\,|0\rangle$

$|n\rangle$ and $|0\rangle$ normalized to unity $\Rightarrow N_n = \frac{1}{\sqrt{n!}}$

(c) $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a+a^+)$

$$\Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & 1 & 0 & 0 & \vdots \\ 1 & 0 & \sqrt{2} & 0 & \vdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \vdots \\ 0 & 0 & \sqrt{3} & 0 & \vdots \\ - & - & - & & \end{pmatrix}$$

(d) $\hat{x}|x\rangle = x|x\rangle \Rightarrow \langle n|\hat{x}|x\rangle = x\langle n|x\rangle \Rightarrow \sum_{n'\in\mathbb{N}} \langle n|\hat{x}|n'\rangle\langle n'|x\rangle = x\langle n|x\rangle$

$\Rightarrow \sum_{n'\in\mathbb{N}} \langle n|x|n'\rangle\, \psi_{n'}(x) = x\,\psi_n(x)$

From (c): $\langle n|x|n'\rangle = \left(\sqrt{n+1}\,\delta_{n,n'-1} + \sqrt{n}\;\;\delta_{n,n'+1}\right)\sqrt{\frac{\hbar}{2m\omega}}$

$\Rightarrow \sqrt{n+1}\,\psi_{n+1} + \sqrt{n}\;\;\psi_{n-1} = x\,\psi_n(x)\sqrt{\frac{2m\omega}{\hbar}}$ (for $n=0$: $\psi_{-1}=0$) (Δ)

(e) Ansatz $\psi_n(x) = 2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}\, h_n\!\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\psi_0(x)$

with $h_0 = 1$ and $h_{n+1}(x) - 2x\,h_n(x) + 2n\,h_{n-1}(x) = 0$ $(\square)$

into (Δ): $\sqrt{n+1}\,2^{-\frac{n+1}{2}}(n+1)!^{-\frac{1}{2}}\, h_{n+1}\!\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\psi_0(x) + \sqrt{n}\;\;2^{-\frac{n-1}{2}}(n-1)!^{-\frac{1}{2}}\, h_{n-1}\!\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\psi_0(x)$

$= x\sqrt{\frac{2m\omega}{\hbar}}\;2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}\, h_n\!\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\psi_0(x)$

After dividing by common factors: $\frac{1}{2}\sqrt{n+1}\,\frac{1}{\sqrt{(n+1)n}}\, h_{n+1}(\xi) + \sqrt{n}\,h_{n-1}(\xi) = \xi\,\frac{\sqrt{2}}{\sqrt{2}\sqrt{n}}\, h_n(\xi)$

define $\xi = \sqrt{\frac{m\omega}{\hbar}}x$

$\Rightarrow h_{n+1}(\xi) - 2\xi\,h_n(\xi) + 2n\,h_{n-1}(\xi) = 0$

as in $(\square)$.

(f) Use $H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2}$ (HW III, [1])

$$H_{n+1}(\xi) = (-1)^{n+1} e^{\xi^2} \frac{d^n}{d\xi^n} (-2\xi) e^{-\xi^2} = (-1)^{n+1} e^{\xi^2} \left(-2n \frac{d^{n-1}}{d\xi^{n-1}} - 2\xi \frac{d^n}{d\xi^n} \right) e^{-\xi^2}$$

$$= -2n H_{n-1}(\xi) + 2\xi H_n(\xi)$$ which is the relation from (3).

Problem [3]

(a) From V.4 in the lecture notes we already know that the radial equation for $R(r)$ in the case $V(\vec r) = 0$ is

$$\left[-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d}{dr} \right) + \frac{\ell(\ell+1)\hbar^2}{2mr^2} \right] R(r) = E\, R(r)$$

Introduce $\rho = \frac{r}{\hbar}\sqrt{2mE} = kr$ with $k = \frac{1}{\hbar}\sqrt{2mE}$ (magnitude of the wave vector of the free particle)

$$\frac{1}{\rho^2} \frac{d}{d\rho}\left(\rho^2 \frac{d}{d\rho} \right) R(\rho) - \frac{\ell(\ell+1)}{\rho^2} R(\rho) + R(\rho) = 0$$

$$\Rightarrow \frac{d^2}{d\rho^2} R + \frac{2}{\rho} \frac{dR}{d\rho} + \left(1 - \frac{\ell(\ell+1)}{\rho^2} \right) R = 0 \qquad (*)$$

(b) Ansatz $j_\ell(\rho) = \frac{\rho^\ell}{2^{\ell+1}\ell!} \int_{-1}^{+1} e^{i\rho s} (1-s^2)^\ell \, ds$ (also known as Poisson's integral representation for sph. Bessel fcts.)

into (*): [dropping common factors]

$$\rho^\ell \int_{-1}^{+1} (is)^2 e^{i\rho s}(1-s^2)^\ell ds + 2\ell\rho^{\ell-1} \int_{-1}^{+1} (is) e^{i\rho s} (1-s^2)^\ell ds + \cancel{\ell(\ell-1)\rho^{\ell-2} \int_{-1}^{+1} e^{i\rho s} (1-s^2)^\ell ds}$$

$$+ 2\rho^{\ell-1} \int_{-1}^{+1} (is) e^{i\rho s} (1-s^2)^\ell ds + \cancel{2\ell\rho^{\ell-2} \int_{-1}^{+1} e^{i\rho s} (1-s^2)^\ell ds} + \rho^\ell \int_{-1}^{+1} e^{i\rho s} (1-s^2)^\ell ds$$

$$\cancel{- \ell(\ell+1)\rho^{\ell-2} \int_{-1}^{+1} e^{i\rho s} (1-s^2)^\ell ds} = 0$$

$$\Rightarrow \rho^\ell \int_{-1}^{+1} e^{i\rho s} (1-s^2)^{\ell+1} ds + \underbrace{2(\ell+1)\rho^{\ell-1} \int_{-1}^{+1} (is) e^{i\rho s} (1-s^2)^\ell ds}_{= -i\rho^{\ell-1} \int_{-1}^{+1} e^{i\rho s} \frac{d}{ds} (1-s^2)^{\ell+1} ds} = 0$$

$$-i\rho^{\ell-1} \int_{-1}^{+1} e^{i\rho s} \frac{d}{ds} (1-s^2)^{\ell+1} ds \underset{\text{part. int.}}{=} \underbrace{e^{i\rho s}(1-s^2)^{\ell+1}\Big|_{-1}^{1}}_{=0} - \int_{-1}^{+1} i\rho\, e^{i\rho s} (1-s^2)^{\ell+1} ds$$

$\Rightarrow 0 = 0$

i.e. the j_ℓ are solutions.

[c] Induction! Check $\ell=0$: $\frac{\sin\rho}{\rho} = \frac{1}{2}\int_{-1}^{+1}\cos(\rho s)\,ds = \frac{1}{2}\int_{-1}^{+1} e^{i\rho s}\,ds = j_0$ ✓

Suppose we know $(-1)^\ell \rho^\ell \left(\frac{d}{\rho d\rho}\right)^\ell \frac{\sin\rho}{\rho} = j_\ell(\rho)$; then for $\ell+1$:

$$j_{\ell+1}(\rho) = \frac{\rho^{\ell+1}}{2^{\ell+2}(\ell+1)!}\int_{-1}^{+1} e^{i\rho s}(1-s^2)^{\ell+1}\,ds \underset{\text{part. int.}}{=} -\frac{\rho^{\ell+1}}{2^{\ell+2}(\ell+1)!}\int_{-1}^{+1}(\ell+1)\frac{-2s}{i\rho}e^{i\rho s}(1-s^2)^\ell\,ds$$

+ boundary term $\propto (1-s^2)^\ell\Big|_{s=\pm1} = 0$

$$= \frac{\rho^\ell}{2^{\ell+1}\ell!}\int_{-1}^{+1}\left(-\frac{d}{d\rho}e^{i\rho s}\right)(1-s^2)^\ell\,ds$$

$$= (-1)^{\ell+1}\rho^\ell \frac{d}{d\rho}\left(\frac{d}{\rho d\rho}\right)^\ell \frac{\sin\rho}{\rho} = (-1)^{\ell+1}\rho^{\ell+1}\left(\frac{d}{\rho d\rho}\right)^{\ell+1}\frac{\sin\rho}{\rho}$$

Explicitly: $j_0(\rho) = \frac{\sin\rho}{\rho}$ $\qquad j_1(\rho) = \frac{\sin\rho}{\rho^2} - \frac{\cos\rho}{\rho}$

$$j_2(\rho) = \frac{3\sin\rho}{\rho^3} - \frac{3\cos\rho}{\rho^2} - \frac{\sin\rho}{\rho}$$

Plots: see attached

[d] Plane wave in z-direction in spherical coordinates:

$$e^{i\vec{k}\cdot\vec{r}} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} C_{\ell,m}\, j_\ell(kr)\, Y_\ell^m(\theta,\phi)$$ (completeness of states in sph. coord.)

$$= \sum_{\ell=0}^{\infty} C_\ell\, j_\ell(kr)\, P_\ell(\cos\theta)$$ (since ϕ does not occur on l.h.s. $m=0$!)

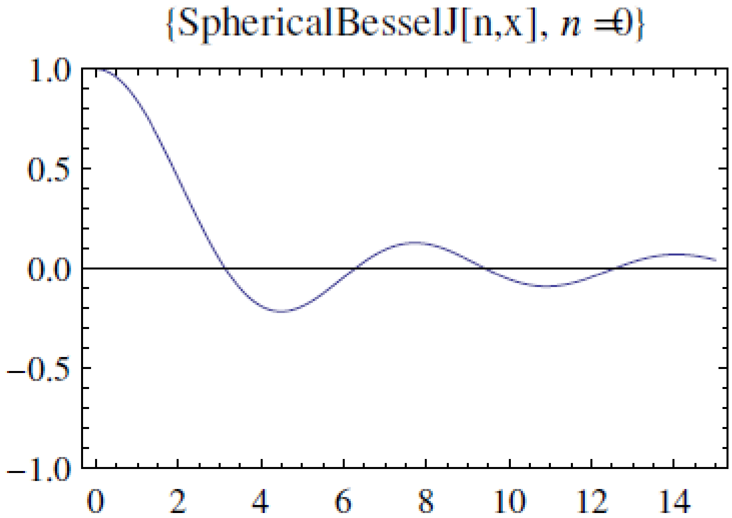
$$\underset{\xi=\cos\theta}{\Longrightarrow} \int_{-1}^{+1} e^{ikr\xi}P_\ell(\xi)\,d\xi = \sum_{\ell'=0}^{\infty} C_{\ell'} j_{\ell'}(kr)\int_{-1}^{+1}P_\ell(\xi)P_{\ell'}(\xi)\,d\xi = \frac{\ell}{2\ell+1}C_\ell j_\ell(kr)$$

$$\Longrightarrow C_\ell = (2\ell+1)\frac{1}{2j_\ell(kr)}\int_{-1}^{+1} e^{ikr\xi}\underbrace{P_\ell(\xi)}_{\frac{(-1)^\ell}{2^\ell \ell!}\frac{d^\ell}{d\xi^\ell}(1-\xi^2)^\ell} \underset{\substack{\text{part. int.}\\ \ell\text{ times}}}{=} (2\ell+1)\frac{1}{j_\ell(kr)}\frac{1}{2^{\ell+1}\ell!}\int_{-1}^{1}\left(\frac{d^\ell}{d\xi^\ell}e^{ikr\xi}\right)(1-\xi^2)^\ell d\xi$$

$\underbrace{\text{+ boundary terms which vanish}}_{= i^\ell j_\ell(kr)}$

$$= (2\ell+1)\,i^\ell$$

$$\Rightarrow e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty}(2\ell+1)\,i^\ell\, j_\ell(kr)\,P_\ell(\cos\theta)$$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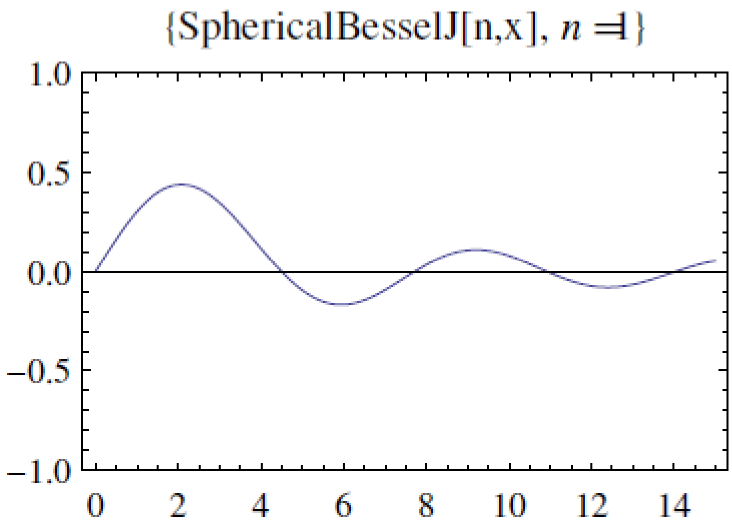
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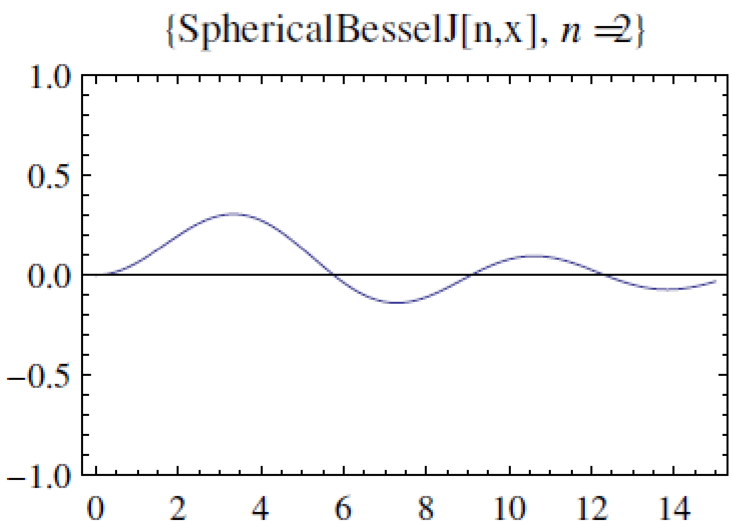
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