PHYS 606 – Spring 2017 – Final Exam Solutions

[1] (a) $|a\rangle = \underbrace{A^{-1}A}_{\mathbb{1}}|a\rangle = \alpha A^{-1}|a\rangle \Rightarrow A^{-1}|a\rangle = \frac{1}{\alpha}|a\rangle$

$\Rightarrow$ Eigenvalue is $\frac{1}{\alpha}$. $\alpha \neq 0$ otherwise A would not be invertible.

(b) $\sum_i |\langle\phi|u|\varphi_i\rangle|^2 = \sum_i \langle\phi|u|\varphi_i\rangle\langle\varphi_i|u^\dagger|\phi\rangle \underset{\text{completeness}}{=} \langle\phi|\underbrace{uu^\dagger}_{\mathbb{1}}|\phi\rangle$

$= \langle\phi|\phi\rangle = 1$

Should have been clear from the start: $|\langle\phi|u|\varphi_i\rangle|^2$ is the probability to find basis state $|\varphi_i\rangle$ in a state $|u^\dagger\phi\rangle$. Sum of these probabilities over a complete basis must be one.

[2] (a) $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\delta(x)$, $V_0 < 0$; Solve $H\psi = E\psi$ for $E<0$

Ansatz: $\psi(x) = \begin{cases} Ae^{-\kappa x} & x>0 \\ Ae^{-\kappa|x|} & x<0 \end{cases}$ for parity-even eigenstates ($[H,\Pi]=0$!)

$\kappa = \frac{1}{\hbar}\sqrt{-2mE}$

$\psi(x) = \begin{cases} Be^{-\kappa x} & x>0 \\ -Be^{-\kappa|x|} & x<0 \end{cases}$ for parity-odd eigenstates

$\psi(x)$ continuous $\Rightarrow$ parity-odd states precluded

Behavior of $\psi'(x)$: Integrate S.E. around $x=0$ in ε-region, let $\varepsilon \to 0$:

$$\int_{-\varepsilon}^{+\varepsilon}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V_0\psi\delta(x)\right)dx = \int_{-\varepsilon}^{\varepsilon} E\,\psi\,dx$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\psi'(\varepsilon)-\psi'(-\varepsilon)\right) + V_0\psi(0) = E\int_{-\varepsilon}^{\varepsilon}\psi(x)dx \to 0 \text{ for } \varepsilon\to 0$$

$$\underset{(\varepsilon\to 0)}{\Rightarrow} 2\psi'(0) = \frac{2mV_0}{\hbar^2}\psi(0) \underset{\text{ansatz}}{\Rightarrow} -\kappa = \frac{mV_0}{\hbar^2} \Rightarrow E = -\frac{mV_0^2}{2\hbar^2}$$

One bound state!

(4) is the free-particle radial equation. solutions = spherical Bessel fcts. $j_\ell(kr)$

(Neumann fcts. not allowed) $\Rightarrow \psi = C' j_\ell(kr) Y_\ell^m(\theta,\phi)$ $\quad k=\frac{1}{\hbar}\sqrt{2mE}$

(b) Matching condition: $j_\ell\left(\frac{a}{\hbar}\sqrt{2mE}\right) = 0 \quad \Rightarrow$ solving this equation gives allowed energies.

$\ell=0: \quad \frac{a}{\hbar}\sqrt{2mE} = n\pi \qquad n=1,2,3,\dots \quad \Rightarrow \quad E_{\ell=0} = \boxed{3.14^2 \frac{\hbar^2}{2ma^2}}, \boxed{6.28^2\frac{\hbar^2}{2ma^2}}, \dots$ (1) (4)

$\ell=1: \quad E_{\ell=1} = \boxed{4.49^2 \frac{\hbar^2}{2ma^2}}, \ 7.73^2 \frac{\hbar^2}{2ma^2}, \dots$ (2)

$\ell=2: \quad E_{\ell=2} = \boxed{5.76^2 \frac{\hbar^2}{2ma^2}}, \dots$ (3)

$\ell=3: \quad \dots$

4 lowest energy eigenvalues are boxed.

[4] One acceptable ansatz: One-parameter Gaussian wave fct.

$$\psi_\lambda(x) = \sqrt[4]{\frac{\lambda}{\pi}}\, e^{-\frac{\lambda x^2}{2}} \quad \text{(properly normalized)}.$$

Solve $\quad \frac{\partial \langle H\rangle[\psi_\lambda]}{\partial\lambda} = 0$

$$\langle H\rangle[\psi_\lambda] = \langle\psi_\lambda|H|\psi_\lambda\rangle = \sqrt{\frac{\lambda}{\pi}}\int e^{-\frac{\lambda x^2}{2}}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - V_0 e^{-\alpha x^2}\right)e^{-\frac{\lambda x^2}{2}}dx$$

$$= \sqrt{\frac{\lambda}{\pi}}\int\left(-\frac{\hbar^2}{2m}(\lambda^2x^2-\lambda) - V_0 e^{-\alpha x^2}\right)e^{-\lambda x^2}dx$$

$$= \sqrt{\frac{\lambda}{\pi}}\left(-\frac{\hbar^2}{2m}\lambda^2\frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}} + \frac{\hbar^2}{2m}\lambda\sqrt{\frac{\pi}{\lambda}} - V_0\sqrt{\frac{\pi}{\lambda+\alpha}}\right) = \frac{\hbar^2}{4m}\lambda - V_0\sqrt{\frac{\lambda}{\lambda+\alpha}}$$

$$\frac{\partial\langle H\rangle[\psi_\lambda]}{\partial\lambda} = \frac{\hbar^2}{2m} - \frac{V_0}{2}\sqrt{\frac{\lambda+\alpha}{\lambda}}\,\frac{(\lambda+\alpha)-\lambda}{(\lambda+\alpha)^2} = \frac{\hbar^2}{4m} - \frac{\alpha V_0}{2}\frac{1}{\sqrt{\lambda(\lambda+\alpha)^3}} \stackrel{!}{=} 0$$

$$\Rightarrow \lambda(\lambda+\alpha)^3 = \frac{4m^2\alpha^2V_0^2}{\hbar^4}$$

probably further numerical processing.

②

b) same with $E>0$

Ansatz $\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x\leq 0) \\ Ce^{ikx} + De^{-ikx} & (x\geq 0) \end{cases}$ $\qquad k=\frac{1}{\hbar}\sqrt{2mE}$

Integration of S.E. gives same condition $\psi'(0^+) - \psi'(0^-) = \frac{2mV_0}{\hbar^2}\psi(0)$ (*)

and smoothness at $x=0$: $A+B = C+D$ (1)

(*) $\Rightarrow -ik(A-B-C+D) = \frac{2mV_0}{\hbar^2}(A+B)$ (2)

(1) $\Rightarrow$ $B = C+D-A$ in (2): $-ik(2A-2C) = \frac{2mV_0}{\hbar^2}(C+D) \Rightarrow A = \frac{imV_0}{\hbar^2 k}(C+D)+C$

$\Rightarrow$ $B = -\frac{imV_0}{\hbar^2 k}(C+D)+D$

For $\begin{pmatrix}A\\B\end{pmatrix} = M\begin{pmatrix}C\\D\end{pmatrix}$ we find $M = \begin{pmatrix} 1+\frac{imV_0}{\hbar^2 k} & \frac{imV_0}{\hbar^2 k} \\ -\frac{imV_0}{\hbar^2 k} & 1-\frac{imV_0}{\hbar^2 k}\end{pmatrix}$

Coefficient of transmission:

$$T = \frac{|C|^2}{|A|^2} = |M_{11}|^{-2} = \frac{1}{1+\frac{m^2V_0^2}{\hbar^4 k^2}}$$

$$R = 1-T = \frac{\frac{m^2V_0^2}{\hbar^4 k^2}}{1+\frac{m^2V_0^2}{\hbar^4k^2}}$$

[3] $\left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{L^2}{2mr^2} + V(r)\right)\psi = E\psi$ (1) ; $L^2\psi = \ell(\ell+1)\hbar^2\psi$ (2) ; $L_z\psi = m\hbar\psi$ (3)

Separation ansatz: $\psi(r,\theta,\phi) = 0$ for $r>a$

$$\psi(r,\theta,\phi) = R(r)\,Y_\ell^m(\theta,\phi) \quad \text{for } r<a$$

$\Rightarrow$ (2),(3) satisfied and $\left[-\frac{\hbar^2}{2m}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right] R = ER$ $\quad (r<a)$ (4)

and $R(a)=0$ (matching condition)

[5] (a) $D_\alpha^+ = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha a^+ - \alpha^* a)^{n^+} = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha a^+ - \alpha^* a)^{+^n} = \sum_{n=0}^{\infty} \frac{1}{n!} (\alpha^* a - \alpha a^+)^n = D_{-\alpha}$

(b) $D_\alpha^+ D_\alpha \underset{(a)}{=} D_{-\alpha} D_\alpha = e^{-(\alpha a^+ - \alpha^* a)} e^{+(\alpha a^+ - \alpha^* a)} \underset{BCH}{=} e^{-(\alpha a^+ - \alpha^* a) + (\alpha a^+ - \alpha^* a)} = e^0 = \mathbb{1}$

(c) $D_\alpha^+ a D_\alpha = D_\alpha^+ \left\{ a \left[\mathbb{1} + (\alpha a^+ - \alpha^* a) + \frac{1}{2!} (\alpha a^+ - \alpha^* a)^2 + \dots \right] \right.$

$= D_\alpha^+ \left[a + (\alpha a^+ a - \alpha^* a + \alpha) + \frac{1}{2!} (\alpha a^+ a - \alpha^* a + \alpha)(\alpha a^+ - \alpha^* a) + \dots \right.$

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$[a, a^+] = \mathbb{1}$

generally $a (\alpha a^+ - \alpha^* a)^n = (\alpha a^+ - \alpha^* a)^n a + n\alpha (\alpha a^+ - \alpha^* a)$

$= D_\alpha^+ \left[a \overset{\alpha +}{+} (\alpha a^+ - \alpha^* a) a + \frac{1}{2!} (\alpha a^+ - \alpha^* a)^2 a + \frac{1}{2!} 2 (a^+ a - \alpha^* a) \alpha + \dots \right]$

$= D_\alpha^+ D_\alpha (a + \alpha) = a + \alpha$

(d) $D_\alpha^+ a D_\alpha |0\rangle \underset{(c)}{=} (a + \alpha)|0\rangle = \alpha |0\rangle$

$\underset{(b)}{\Rightarrow} \quad a D_\alpha |0\rangle = \alpha D_\alpha |0\rangle$