Analytical Mechanics

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Chapter 1

Review of Newtonian Mechanics

In mechanics we study the motion of

- point masses
- rigid bodies (= systems of point masses with additional constraints on their relative spatial locations)
- media (elastic bodies, fluids, etc.)

under certain conditions which allow the laws of mechanics to be valid. More precisely we demand relative velocities are much smaller than the speed of light, $v \ll c$, and increments of the action variable be much larger than Planck's constant, $\Delta S \gg h$, in order to keep corrections from special relativity and quantum mechanics small.

Of course, with large numbers of masses we sometimes can only hope to make statistical statements about their motion which leads to the separate topic of statistical mechanics. Until further notice (basically up to chapter 4) we will always deal with point masses or extended objects *approximated* by point masses.

1.1 Fundmental Principles of Classical Mechanics

It turns out that mechanics can be based on a few axioms from which all equations of motion can be derived. Here we choose the following set.

(A) Our space is 3-dimensional and euclidean, and time is one-dimensional.

(B) Galileo's Principle of Relativity: There exist coordinate systems (called "inertial") with the following properties: (i) All laws of classical mechanics at all moments of time are the same in all inertial coordinate systems. (ii) All coordinate systems in uniform, rectilinear motion with respect to an inertial one are themselves inertial.

(C) Newton's Principle of Determinacy: The initial state of a mechanical system, i.e. the totality of positions and velocities of its point masses at some moment in time, *uniquely* determines all of its motion.

We will later add two more such statements which will allow us to connect motion to the concepts of force and mass.

1.2 Kinematics

1.2.1 Basic Definitions

□ We consider the configuration space for n particles[1] $\mathbb{R}^N = (\mathbb{R}^3)^n$ as the direct product of the 3-dimensional spaces of positions for each particle. A motion in the configuration space of this system is a differentiable mapping $x : I \to \mathbb{R}^N$, $t \mapsto x(t)$. $I \subset \mathbb{R}$ is an interval on the real line with t representing time and $x(t)$ is the position of the system in configuration space at time t. We read $x = (\vec{x}_1, \ldots, \vec{x}_n)$ where $\vec{x}_i$ is the position of particle i.

□ The derivative $v(t) = \dot{x}(t) = dx/dt \in \mathbb{R}^N$ is called the velocity at time t. Again $v = (\vec{v}_1, \ldots, \vec{v}_n)$ where the $\vec{v}_i$ are the velocity vectors of each particle. Similarly, we define the acceleration at time t as $a(t) = \dot{v}(t) = d^2x/dt^2$.

□ The *image* of the mapping $x(t)$ in $\mathbb{R}^N$ is called the *trajectory* of the mechanical system. The *graph* of the mapping $x(t)$ in $I \times \mathbb{R}^N$ is called the *world line* of the mechanical system. For $n > 1$ trajectories or world lines are often shown as a family of trajectories or world lines of individual particles in $\mathbb{R}^3$ or $I \times \mathbb{R}^3$ resp.

1.2.2 Trajectories of Single Point Masses

Let the number of particles be $n = 1$ in this subsection.

□ Given the motion $\vec{x}(t)$ we can easily calculate the length l of the trajectory between two points in time t_i and t_f. We divide the time interval into k segments of width $\Delta t = (t_f - t_i)/k$ and approximate the trajectory by straight segments between neighboring points $\vec{x}(t_i)$ and $\vec{x}(t_{i+1})$ for $i = 1, \ldots, k$ (we take $t_1 = t_i$ and $t_{k+1} = t_f$). The length l_k of the chain of straight segments is

$$l_k = \sum_{i=1}^{k} |\vec{x}(t_{i+1}) - \vec{x}(t_i)| = \sum_{i=1}^{k} \left| \frac{\vec{x}(t_{i+1}) - \vec{x}(t_i)}{\Delta t} \right| \Delta t \,. \tag{1.1}$$

Obvioulsy $\lim_{k\to\infty} l_k = l$ and thus

$$l = \int_{t_i}^{t_f} dt \left| \frac{d\vec{x}}{dt} \right| , \tag{1.2}$$

i.e. the length is given by the time integral of the speed.

□ We can define a variable s that counts the distance covered from the start of the trajectory at time t_0 through a function $s : I \to \mathbb{R}$,

$$s(t) = \int_{t_0}^{t} dt' \left| \frac{d\vec{x}}{dt'} \right| . \tag{1.3}$$

Obviously $ds/dt = |d\vec{x}/dt| > 0$, i.e. s is a monotonic and invertible function of time. We can reparameterize a trajectory in a way that is independent of how a point mass moves along by replacing $\vec{x}(t)$ by $\vec{x}(s)$ such that $\vec{x}(t) = \vec{x}(s(t))$. $\vec{x}(s)$ is called the natural parameterization of the trajectory.

□ Fundamental calculus shows that $d\vec{x}/dt$ is tangential to the trajectory at time t pointing into the future. Hence we can define a unit tangential vector (called *the* tangent vector)

$$\hat{t}(t) = \frac{d\vec{x}/dt}{|d\vec{x}/dt|} \,. \tag{1.4}$$

[1] *Particle* is often used instead of *point mass* in this manuscript.

Figure 1.1: A trajectory with torsion shown with is osculating plane at the point $\vec{x}(s_0)$.

for any point along the trajectory. Note that if the trajectory is given in natural parameterization

$$\hat{t}(s) = d\vec{x}/ds\,. \tag{1.5}$$

Proof: $\hat{t} = (d\vec{x}/dt)\,/\,(ds/dt) = (d\vec{x}/dt)\,(dt/ds) = d\vec{x}/ds$.

□ The absolute value of the derivative of $\hat{t}$ with respect to s

$$\kappa = \left|\frac{d\hat{t}}{ds}\right| \tag{1.6}$$

is called the *curvature* of the trajectory at a point (parameterized by) s. $\rho = \kappa^{-1}$ is called the radius of curvature.

□ Note that $d\hat{t}/ds \perp \hat{t}$. In general, for *any* unit vector as a function of a single parameter the derivative vector is orthogonal to the vector itself for any value of the parameter. Proof: $d\hat{t}/ds \cdot \hat{t} = 1/2\, d/ds\left(\hat{t}\cdot\hat{t}\right) = 0$.

□ Hence the vector

$$\hat{n}(s) = \frac{d\hat{t}/ds}{|d\hat{t}/ds|} = \rho\frac{d\hat{t}}{ds} \tag{1.7}$$

is a unit vector orthogonal to $\hat{t}$. It is called the *normal* vector at the point s.

□ $\hat{t}$ and $\hat{n}$ span the *osculating* or "kissing" plane at a given point of the trajectory.

$$\hat{b} = \hat{t}\times\hat{n} \tag{1.8}$$

is called the binormal unit vector; it is orthogonal to the osculating plane. In particular, $\hat{b}(s) =$ const. $\Longleftrightarrow$ motion is confined to the osculating plane.

□ We have

$$\frac{d\hat{b}}{ds} = \hat{t}\times\frac{d\hat{n}}{ds} \tag{1.9}$$

Proof: Work it out as a homework problem.

Hence we can write

$$\frac{d\hat{b}}{ds} = -\tau\hat{n} \tag{1.10}$$

at any point since $d\hat{b}/ds$ is orthogonal to both $\hat{t}$ and $\hat{b}$. $\tau(s)$ is called the *torsion* of the trajectory at the point given by s and $\sigma = \tau^{-1}$ is the radius of torsion.

□ We have

$$\frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}\,. \tag{1.11}$$

Proof: Work it out as a homework problem.

□ Obviously $\hat{t}(s)$, $\hat{n}(s)$ and $\hat{b}(s)$ define a set of mutually orthogonal unit vectors with positive orientiation, at any point s along the trajectory. I.e. they are a comoving coordinate system. The identities for their derivatives

$$\boxed{\frac{d\hat{t}}{ds} = \kappa\hat{n} \qquad \frac{d\hat{b}}{ds} = -\tau\hat{n} \qquad \frac{d\hat{n}}{ds} = \tau\hat{b} - \kappa\hat{t}} \tag{1.12}$$

are called the *Frenet-Serret-Formulas*.

☐ Now we can write for the velocity

$$\vec{v}(t) = \frac{ds}{dt}\hat{t} \tag{1.13}$$

with $v = ds/dt$ called the speed. In particular $\vec{v}$ is tangential to the trajectory. Proof: $\vec{v} = (d\vec{x}/ds)(ds/dt) = ds/dt\,\hat{t}$.

☐ For the acceleration we have

$$\vec{a}(t) = \dot{v}\hat{t} + \frac{v^2}{\rho}\hat{n} \tag{1.14}$$

where v is the speed. I.e. the acceleration has a tangential and a normal component but is always confined to the osculating plane. Proof: Work it out as a homework problem.

☐ Example: Consider a circular motion with constant angular acceleration $\alpha = \ddot{\phi}$ and radius R in the center of the x_1-x_2-plane. Then we have $\vec{x}(t) = R(\cos\alpha t^2/2, \sin\alpha t^2/2, 0)$ and obviously $\vec{v}(t) = \alpha t R(-\sin\alpha t^2/2, \cos\alpha t^2/2, 0)$. We can switch to natural parameterization by considering $|v(t)| = \alpha t R$ and $s(t) = \int_0^t |v| dt' = \alpha t^2 R/2$. Hence we arrive at $\vec{x}(s) = R(\cos s/R, \sin s/R, 0)$ and $\vec{v}(s) = R(-\sin s/R, \cos s/R, 0)$. It is easy to see that

$$\hat{t}(s) = (-\sin s/R, \cos s/R, 0) \tag{1.15}$$

$$\hat{n}(s) = (-\cos s/R, -\sin s/R, 0) \tag{1.16}$$

$$\hat{b}(s) = (0, 0, 1) \tag{1.17}$$

while the radius of curvature is $\rho = R$. The second equation follows from the first Frenet-Serret-Formula. We can now specify the components of the acceleration vector. The tangential acceleration is $\dot{v} = \alpha R$ while the normal acceleration is $v^2/\rho = \alpha^2 t^2 R$.

1.3 Galilei Transformations

☐ We now would like to classify the coordinate transformations that are allowed according to axiom (B) in Sec. 1.1. Transformations $\mathcal{G}$ from one inertial system K into another inertial system K' are called Galilean transformations. We discuss them as applied to the time t, spatial coordinates $\vec{x}$ and velocity v of a single particle. It is understood that for more than one particle the same transformation rules have to be repeated for all particles.

1. Rotations and spatial inversions: $\vec{x} \mapsto \mathbf{R}\vec{x}$ (with t, $\vec{v}$ kept constant) where $\mathbf{R}$ is an orthogonal 3×3-matrix, i.e. a member of the group O(3). Note that all matrices in this group have either $\det\mathbf{R} = +1$ (they form a subgroup called SO(3)), or $\det\mathbf{R} = -1$. The latter can always be written as a product of a member of SO(3) and an inversion $\vec{x} \mapsto -\vec{x}$. The group SO(3) is the group of rotations in 3 dimensions, all members of the group can be parameterized by 3 rotation angles (more details on how to do that later). I.e. we have 3 parameters to specify rotations.

2. Uniform motion: $\vec{v} \mapsto \vec{v} + \vec{w}$, $\vec{x} = \vec{x} + \vec{w}t$ (with t constant). $\vec{w}$ is a constant velocity which means that we have 3 parameters for this kind of transformation.

3. Translations in space: $\vec{x} \mapsto \vec{x} + \vec{a}$ (with t, $\vec{v}$ constant). $\vec{a}$ is a constant vector, hence there are 3 parameters for translations.

4. Translations and inversions in time: $t \mapsto \lambda t + b$, $\vec{v} \mapsto \lambda\vec{v}$ ($\vec{x}$ constant). Here b is an arbitrary constant and $\lambda = \pm 1$. b counts as one new continuous parameter.

□ The most general Galilei transformation hence has the shape

$$t \mapsto \lambda t + b \qquad \vec{x} \mapsto \mathbf{R}\vec{x} + \vec{w}\lambda t + \vec{a} \qquad \vec{v} \mapsto \lambda\vec{v} + \vec{w} \tag{1.18}$$

Galilei transformations form a group. We restrict our discussion to the part of the group continuously connected to the identity, which means that we fix $\det \mathbf{R} = +1$ and $\lambda = +1$. This is the group of *proper, orthochronous Galilei transformations*. Its elements are described by 10 continuous parameters.
□ Later: symmetries ⇔ conservation laws (under certain conditions).
□ Example: Consider the acceleration on a satellite in earth's orbit. According to Newton's Law of Gravity $\ddot{x} = Gm_E/x^2$ where x is the distance to the center of earth, m_E is earth's mass and G is the graviational constant. This acceleration is obviously not translationally invariant: $Gm_E/x^2 \mapsto Gm_E/|\vec{x} + \vec{a}|^2$. Newton's Law of Gravity is rather fundamental, does it violate axiom (B)? Answer: In this form yes, the presence of earth brakes translational invariance by designating a particular point in space as the origin of the coordinate system. Earth is "external" to the system described. However, we can make earth part of the system and write down Newton's Law of Gravity for the satellite and earth at $\vec{x}_S$ and $\vec{x}_E$ resp.: $d^2(\vec{x}_S - \vec{x}_E)/dt^2 = G\mu/|\vec{x}_E - \vec{x}_S|^2$ where μ is the reduced mass of the earth-satellite system. (Check this with your textbook on gravity!) Now a translation $\vec{x} \mapsto \vec{x} + \vec{a}$ leaves the law invariant since only differences of position vectors enter!
□ We can generalize the previous example: the existence of external conditions, e.g. external forces applied to the system, usually breaks Galilei invariance. However, if we make the system sufficiently large such that all forces are "internal" the system must exhibit Galilean invariance.

1.4 Newton's Laws

□ According to axiom (C) the motion of a system is uniquely determined by the positions $x(t_0)$ and velocities $\dot{x}(t_0)$ of a system at any time t_0 (the *initial conditions*). Hence also the acceleration at time t_0, $\ddot{x}(t_0)$ is determined by $x(t_0)$, $\dot{x}(t_0)$ and t_0 Since the argument applies to any t_0 there must exist a function $A : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, with $(x, \dot{x}, t) \mapsto A(x, \dot{x}, t)$ such that

$$\boxed{\ddot{x} = A(x; \dot{x}; t)} \tag{1.19}$$

□ In other words, there is a second order differential equation for the motion $x(t)$. This is known as Newton's Equation for the system. Mathematically, A defines the mechanical system (it contains the forces to be discussed further down). A and the initial conditions together uniquely determine the motion of the system.

1.4.1 Newton's Equation for Systems with Galilei Invariance

□ Consider a system with positions $x = (\vec{x}_1, \ldots, \vec{x}_n)$, described by an acceleration $A(x, \dot{x}, t)$. We investigate the constraints on A if the following symmetries hold:

1. Time invariance ⇒ $A(x; \dot{x}; t) = A(x; \dot{x})$, i.e. no explicit time dependence.
2. Translational invariance ⇒ $A(\vec{x}_1, \ldots \vec{x}_n; \dot{\vec{x}}_1, \ldots, \dot{\vec{x}}_n; t) = A(\vec{x}_i - \vec{x}_j, i, j = 1, \ldots n; \dot{\vec{x}}_1, \ldots, \dot{\vec{x}}_n; t)$, i.e. A can only depend on differences between positions (cf. example in Sec. 1.3).

3. Invariance under transformations with constant velocity $\Rightarrow A(\vec{x}_1, \dots \vec{x}_n; \dot{\vec{x}}_1, \dots, \dot{\vec{x}}_n; t) = A(\vec{x}_1, \dots \vec{x}_n; \dot{\vec{x}}_i - \dot{\vec{x}}_j, i, j = 1, \dots n; t)$, i.e. A can only depend on differences between velocities of particles.

4. Isotropy $\Rightarrow A(\mathbf{R}\vec{x}, \mathbf{R}\dot{\vec{x}}, t) = \mathbf{R}A(x, \dot{x}, t)$ for all $\mathbf{R} \in \mathrm{SO}(3)$. Obviously we define $\mathbf{R}$ to act on an element of configuration space $\mathbb{R}^N$ by applying it blockwise, i.e. $\mathbf{R}x = (\mathbf{R}\vec{x}_1, \dots, \mathbf{R}\vec{x}_n)$.

☐ Full Galilei Invariance (1)-(4) puts tight constraints on A.
☐ Example: For a single point mass in an inertial frame we have $\vec{A} = \vec{0}$. This is known as Newton's First Law. Proof: According to (2.) and (3.) from the last section $\vec{A}$ only depends on differences between coordinates and velocities. Since there is only one position and one velocity $\vec{A}$ must be independent of those, according to (1) it also can not depend on t. Hence $\vec{A}(\vec{x}, \dot{\vec{x}}, t) = \vec{A} =$ const. (4.) requires $\vec{A}$ to be invariant under rotations, $\mathbf{R}\vec{A} = \vec{A}$ for any $\mathbf{R}$. The only vector with that property is $\vec{A} = \vec{0}$. Note: The presence of any "external" force acting on the point mass would break Galilei invariance.

1.4.2 Forces and Masses

☐ To analyze mechanical systems it is useful to break $\vec{A}$ for a particle into a part that depends only on the particle, and a part that depends on "other" parts of the system or external forces. Indeed $\vec{A}$ factorizes in the following way:

(D) We can attach a property m to a particle (the *inertial mass* such that two otherwise equal particles with masses m_1 and m_2 in the same system with the same positions and velocities at a time t_0 have accelerations at t_0 behaving like

$$\frac{\ddot{\vec{x}}_1}{\ddot{\vec{x}}_2} = \frac{m_2}{m_1}\,. \tag{1.20}$$

☐ We call

$$\vec{F} = m\ddot{\vec{x}} = m\vec{A}(\vec{x}, \dot{\vec{x}}, t) \tag{1.21}$$

the force (field) acting on the particle.
☐ We have two more established experimental facts:

(E) Superposition principle: If a system is assembled in which a particle of mass m is subject to several forces $\vec{F}_i$ the motion is described by the linear superposition of these forces, i.e.

$$\vec{A} = \frac{1}{m}\sum_i \vec{F}_i,\ . \tag{1.22}$$

(F) Newton's Third Law: In a system with at least two particles a force $\vec{F}_{ij}$ from particle j onto particle i is reciprocated by a force $\vec{F}_{ji} = -\vec{F}_{ij}$ from particle i onto particle j.

☐ For a single point mass of mass m and velocity $\vec{v}$ we define the *momentum* as

$$\vec{p} = m\vec{v}\,. \tag{1.23}$$

1.4.3 Summary of Newton's Laws of Mechanics

☐ From the previous subsections it follows immediately that

(I) An object in uniform motion tens to remain in uniform motion unless an external force is applied.

(II) In an inertial coordinate system with a force $\vec{F}$ acting on a mass m the change in momentum is

$$\boxed{\dot{\vec{p}} = \vec{F}} \tag{1.24}$$

(III) For systems with $n > 1$ particles we have $\vec{F}_{ij} = -\vec{F}_{ji}$.

☐ Mathematically, for a system with N degrees of freedom Newton's Second Law leads to a system of 2nd order differential equations. The solution $x(t)$ depends on $2N$ parameters which are fixed by initial conditions to determine a unique motion.

1.4.4 Important Forces

☐ A force $\vec{F}(\vec{r})$ is called a *central force* if it points to or from a center, i.e.

$$\vec{F} = F(r)\hat{r} \tag{1.25}$$

if the center is at the origin. Here $\hat{r}$ is the unit vector belonging to $\vec{r}$ and $r = |\vec{r}|$. See Sec. 1.3 on how to write this as a force between two particles that respects Galilei Invariance.

☐ Examples for central forces:

- Newton's Law of Gravity

$$\vec{F}_{12} = G\frac{m_1 m_2}{r^3}(\vec{r}_1 - \vec{r}_2) \tag{1.26}$$

- Coulomb Force

$$\vec{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 r^3}(\vec{r}_1 - \vec{r}_2) \tag{1.27}$$

☐ Lorentz force on a charge q in electric and magnetic fields $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) . \tag{1.28}$$

☐ Elastic force or spring (Hooke's Law):

$$\vec{F} = -k\vec{x} \tag{1.29}$$

where x is the displacement from equilibrium and k is the spring constant

1.5 Conservation Laws

1.5.1 Work and Energy

□ Consider a force $\vec{F}(\vec{x}, \dot{\vec{x}}, t)$ acting on a point mass moving along a trajectory $\mathcal{C}$. The*work* done by $\vec{F}$ along $\mathcal{C}$ is given by the contour integral

$$W_{\mathcal{C}} = \int_{\mathcal{C}} \vec{F} \cdot d\vec{s} \tag{1.30}$$

where $d\vec{s} = \hat{t}ds$ and ds is a line element along the trajectory.
□ A note on calculating the work integral. Even though we call $\mathcal{C}$ a trajectory we do need the information on the whole motion, i.e. $\vec{x}(t)$. In that case we can write $\vec{F} \cdot d\vec{s} = \vec{F}(\vec{x}(t), \dot{\vec{x}}(t), t) \cdot \dot{\vec{x}}\, dt$.
□ A force field $\vec{F}$ is called conservative if there exists a scalar field $U(\vec{x})$ such that

$$\vec{F} = -\mathrm{grad}\, U = -\nabla U\,. \tag{1.31}$$

Here $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. Note that U is determined by $\vec{F}$ only up to a constant. U is called a potential energy (function) for $\vec{F}$. In particular: a conservative force can not depend on velocities and it can not depend explicitly on time: $\vec{F}(\vec{x}, \dot{\vec{x}}, t) = \vec{F}(\vec{x})$.
□ For a force field $\vec{F}(\vec{x})$ on a simply connected subspace[2] of $\mathbb{R}^3$ the following conditions are equivalent:

1. $\vec{F}(\vec{x}) = -\mathrm{grad}\, U(\vec{x})$ for all $\vec{x}$
2. $\mathrm{curl}\, \vec{F}(\vec{x}) = \vec{0}$ for all $\vec{x}$.
3. $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{s} = 0$ for all closed loops $\mathcal{C}$.
4. For any arbitrary $\mathcal{C}$ the work integral $\int_{\mathcal{C}} \vec{F} \cdot d\vec{s}$ only depends on the endpoints P_1 and P_2 of $\mathcal{C}$.

Note that the curl is sometimes written as $\mathrm{curl}\, \vec{F} = \nabla \times \vec{F}$.
□ Several of the equivalencies are proven quite easily. E.g. 3. ⇔ 4. is simple, and 1. ⇒ 2. follows from basic vector calculus. We would like to show a proof for 1. ⇒ 4.. Let $\mathcal{C}$ and $\mathcal{C}'$ be two arbitrary trajectories with the same endpoints P_1 and P_2. Then

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s} = -\int_{\mathcal{C}} \mathrm{grad}\, U \cdot \frac{d\vec{x}}{dt} dt = -\int_{\mathcal{C}} \frac{dU}{dt} dt = -\int_{\mathcal{C}} dU = -\left(U(P_2) - U(P_1)\right) \tag{1.32}$$

Obviously the result for $\mathcal{C}'$ is the same, q.e.d.
□ We conclude: for a conservative force $\vec{F}(\vec{x})$ work only depends on the beginning and end point of a trajectory and

$$W_{\mathcal{C}} = W_{21} = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{s} = U(P_1) - U(P_2)\,. \tag{1.33}$$

□ Examples for non-conservative forces: time-dependent forces, friction.

[2]This is a requirement on the topology of the subspace. Basically it is not allowed to have any holes; in other words for any closed loop on this space there should be a continuous map that contracts the loop into a single point.

☐ Central forces are conservative. Because of rotational symmetry we have $U(\vec{r}) = U(r)$ in that case with $r = |\vec{r}|$. Proof:

$$\operatorname{curl} \vec{F} = \frac{F(r)}{r} \operatorname{curl} \vec{r} + \operatorname{grad} \frac{F(r)}{r} \cdot \vec{r} = \vec{0} \,. \tag{1.34}$$

For the second term, recall that lines of constant value of $F(r)/r$ are circles around the origin and the gradient of the field is perpendicular to those lines.

☐ Simple examples:

- $1/r^2$ central force (Gravity, Coulomb): Assume $\vec{F}(\vec{r}) = -\alpha/r^2 \hat{r}$. Choose the reference point P_1 with vanishing U at infinity. Then

$$U(\vec{r}) = -\int_{\infty}^{\vec{r}} \left(-\frac{\alpha}{r'^2}\hat{r}\right) \cdot (\hat{r} dr') = \alpha \int_{\infty}^{r} \frac{dr'}{r'^2} = -\frac{\alpha}{r} \,. \tag{1.35}$$

- Elastic force (Hooke's Law): $\vec{F} = -k\vec{x} \Rightarrow U(\vec{x}) = \frac{1}{2}kx^2$.

1.5.2 Energy Conservation

☐ For a point mass m with velocity $\vec{v}$ we define the kinetic energy

$$T = \frac{1}{2} m v^2 \,. \tag{1.36}$$

☐ Consider a point mass moving from P_1 to P_2 along a trajectory $\mathcal{C}$. Let $\vec{F}(\vec{x}, \dot{\vec{x}}, t)$ be the total force on the point mass. Then we have

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{s} = T(P_2) - T(P_1) \,, \tag{1.37}$$

i.e. the change in kinetic energy is equal to the work done by $\vec{F}$. This is the *Work-Energy-Theorem*. Note that it is also valid for non-conservative forces. Proof: Multiplying Newton's Second Law with the velocity we integrate over time along the trajectory:

$$m\ddot{\vec{x}} \cdot \dot{\vec{x}} = \vec{F} \cdot \dot{\vec{x}} \tag{1.38}$$

$$\Rightarrow \int_{t(P_1)}^{t(P_2)} \frac{d}{dt}\left(\frac{m}{2}\dot{\vec{x}}^2\right) dt = \int_{t(P_1)}^{t(P_2)} \vec{F} \cdot \dot{\vec{x}} dt \tag{1.39}$$

$$\Rightarrow T(P_2) - T(P_1) = W_{\mathcal{C}} \tag{1.40}$$

☐ For a conservative force $\vec{F}(\vec{x})$ we define the energy of a point mass subject to this force as $E = T + U$. Applying the Work-Energy-Theorem to two points P_1, P_2 along a trajectory $\mathcal{C}$ gives

$$T(P_2) - T(P_1) = W_{\mathcal{C}} = -U(P_2) + U(P_1) \Rightarrow T(P_1) + U(P_1) = T(P_2) + U(P_2) \,. \tag{1.41}$$

Hence $E(P_1) = E(P_2)$, or in other words the energy along the trajectory is constant. This is the law of energy conservation for conservative forces.

☐ For a system of n particles we define the kinetic, potential, and total energy of the system as the sum of the individual kinetic, potential, and total energies. Potential energies coming from mutual forces $\vec{F}_{ij} = -\vec{F}_{ji}$ between two particles are counted only once. In other words

$$U(\vec{x}_1, \ldots, \vec{x}_n) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} U_{ij}(\vec{x}_i - \vec{x}_j) = \frac{1}{2} \sum_{i,j=1}^{n} U_{ij}(\vec{x}_i - \vec{x}_j)\,, \tag{1.42}$$

where U_{ij} is a potential energy with $\vec{F}_{ij} = -\nabla_i U_{ij}$. Note that Newton's Third Law requires U_{ij} to depend only on the difference in position vectors, $\vec{x}_i - \vec{x}_j$. It turnes out that this is the correct definition to prove the following generalization of the law of energy conservation for systems of particles.

☐ Consider a system of n particles with masses m_i with conservative forces $\vec{F}_{ij} = -\vec{F}_{ji}$ acting between particles i and j. Each particle i is also subject to an external force $\vec{F}_i^{(\text{ext})}$ External means that each of these forces does not depend on the positions or velocities of particles other than i. We have

$$\frac{d}{dt}(T + U) = \sum_{i=1}^{n} \vec{v}_i \cdot \vec{F}_i^{(\text{ext})}\,, \tag{1.43}$$

where U is the potential energy related to internal forces and hence $E = T + U$ is the "internal" energy of the system. $\vec{v} \cdot \vec{F}$ is called the power P of the force $\vec{F}$ and the term on the right hand side is equal to the total external power $P^{(\text{ext})}$ added to or subtracted from the system. Hence in other words

$$\boxed{\frac{dE}{dt} = P^{(\text{ext})}}\,. \tag{1.44}$$

Proof: See HW 1.

☐ Example: If the external forces are friction forces the last theorem reads: the rate of change in total energy of a system is given by the power dissipated by friction (obviously $P^{(\text{diss})} < 0$).

1.5.3 Momentum Conservation

☐ For a single point mass $\vec{p} = \text{const.} \Leftrightarrow \vec{F} = \vec{0}$. This is Newtons's First Law.

☐ Consider a system of n particles with masses m_i with forces $\vec{F}_{ij} = -\vec{F}_{ji}$ acting between particles i and j. Each particle i is also subject to an external force $\vec{F}_i^{(\text{ext})}$. Let $\vec{p}_i = m_i \vec{v}_i$ denote the momentum of each particle. Then we define the total momentum of the system as

$$\vec{P} = \sum_{i=1}^{n} \vec{p}_i \tag{1.45}$$

and we have

$$\boxed{\dot{\vec{P}} = \vec{F}^{(\text{ext})}} \tag{1.46}$$

where $\vec{F}^{(\text{ext})} = \sum_i \vec{F}_i^{(\text{ext})}$ is the total external force. Proof: This follows in a straightforward way from Newton's Second Law since the internal forces cancel pairwise due to Newton's Third Law.

☐ We conclude that the total momentum of a system is conserved exactly if the sum of external forces vanishes. This is the law of conservation of momentum.

☐ We define the center of mass $\vec{X}$ for a system of point masses as

$$\vec{X} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{x}_i \tag{1.47}$$

where $M = \sum_{i=1}^{n} m_i$ is the total mass of the system.

□ The motion of the center of mass is determined by the sum of external forces

$$M\ddot{\vec{X}} = \vec{F}^{(\mathrm{ext})} \,. \tag{1.48}$$

Hence we can describe the motion of the center of mass of extended objects as that of a point mass with mass M with all external forces acting on it.

1.5.4 Angular Momentum Conservation

□ Consider a point mass m with momentum $\vec{p}$ and a force $\vec{F}$ acting on it. Choose a fixed reference point P (at position $\vec{x}_P$) and let $\vec{r}$ be the vector from the reference point to the point mass. We define the torque $\vec{M}$ on the mass and the angular momentum $\vec{L}$ of the mass with respect to the reference point P as

$$\vec{M} = \vec{r} \times \vec{F} \,, \tag{1.49}$$

$$\vec{L} = \vec{r} \times \vec{p} \,, \tag{1.50}$$

respectively. Note that $\vec{M}$ and $\vec{L}$ depend on the choice of reference point.

□ For a single point mass with $\vec{F}$ being the only force acting we have

$$\frac{d}{dt}\vec{L} = \vec{M} \,. \tag{1.51}$$

In particular, the angular momentum of a particle (with respect to P) is conserved exactly if the torque on that particle (with respect to the same P) vanishes. Proof: Taking the vector product of $\vec{r}$ with Newton's Second Law we get

$$m\vec{r} \times \ddot{\vec{x}} = \vec{r} \times \vec{F} = \vec{M} \,. \tag{1.52}$$

We insert $0 = \dot{\vec{x}} \times \dot{\vec{x}} - \dot{\vec{x}}_p \times \dot{\vec{x}} = \dot{\vec{r}} \times \dot{\vec{x}}$ on the right hand side to get

$$\vec{M} = m\left(\dot{\vec{r}} \times \dot{\vec{x}} + \vec{r} \times \ddot{\vec{x}}\right) = m\frac{d}{dt}\left(\vec{r} \times \dot{\vec{x}}\right) \,. \tag{1.53}$$

□ We note that $\vec{L} =$ const. with respect to one reference point does not mean that $\vec{L}$ is constant for other reference points. In general, for $\vec{r} \mapsto \vec{r} + \vec{a}$ we have $\vec{M} \mapsto \vec{M} + \vec{a} \times \vec{F}$.

□ For an object with mass m subject to a central force field $\vec{F}$ the angular momentum $\vec{L}$ w.r.t. the center of the force is conserved. Proof: $\vec{M} = \vec{r} \times (F(r)\hat{r}) = \vec{0}$.

□ If a mass m moves with angular momentum $\vec{L}$ conserved w.r.t a point P then the motion takes place in the plane perpendicular to $\vec{L}$ which contains P. Proof: Let us put the center of the coordinate system in P. Then $\vec{r}$ is the position of the mass and $\vec{r} \cdot \vec{L} = \vec{r}(\vec{r} \times \vec{p}) = 0$.

□ In the above situation the rate at which the "radius" vector $\vec{r}$ from P to the mass sweeps over an area S in that plane is constant, i.e. $dS/dt =$ const. This is also known as Kepler's Second Law. Proof: The area swept out between times t and $t + dt$ is

$$dS = \frac{1}{2}\left|\vec{r}(t) \times \vec{r}(t + dt)\right| = \frac{1}{2}\left|\vec{r}(t) \times \dot{\vec{r}}(t)\right| dt \,. \tag{1.54}$$

Hence $dS/dt = |\vec{L}|/(2m)$.

☐ Consider a system of n particles with masses m_i with forces $\vec{F}_{ij} = -\vec{F}_{ji}$ acting between particles i and j along the vector $\vec{r}_{ij}$ pointing from j to i. Each particle i is also subject to an external force $\vec{F}_i^{(\text{ext})}$ We define the total angular momentum of the system w.r.t. a reference point P as

$$\vec{L} = \sum_{i=1}^{n} \vec{L}_i = \sum_{i=1}^{n} m_i \vec{r}_i \times \dot{\vec{r}}_i \,, \tag{1.55}$$

and we have

$$\boxed{\dot{\vec{L}} = \vec{M}^{(\text{ext})}} \tag{1.56}$$

where $\vec{M}^{(\text{ext})} = \sum_i M_i^{(\text{ext})} = \sum_i \vec{r}_i \times F_i^{(\text{ext})}$ is the total external torque. I.e. the total angular momentum of a system with respect to a reference point P is conserved exactly if the total external torque vanishes. Proof: We have

$$\dot{\vec{L}} = \sum_{i=1}^{n} m_i \left(\vec{r}_i \times \ddot{\vec{r}}_i\right) = \sum_{i,j=1}^{n} \vec{r}_i \times \vec{F}_{ij} + \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i^{(ext)} = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \vec{r}_{ji} \times \vec{F}_{ji} + \sum_{i=1}^{n} M_i^{(\text{ext})} \tag{1.57}$$

☐ For a closed system of particles with conservative forces the total energy, total momentum and total angular momentum are conserved and the center of mass moves as

$$\vec{X}(t) = \frac{1}{M}\vec{P}t + \vec{X}(0)\,. \tag{1.58}$$

1.5.5 Two-Particle Systems

☐ Consider two point masses m_1, m_2 at positions $\vec{r}_1$, $\vec{r}_2$, respectively. As usual there can be external forces on each of the masses and internal forces $\vec{F}_{12} = -\vec{F}_{21}$ betweeen them. Instead of using $\vec{r}_1$, $\vec{r}_2$, we can describe the system by using the center of mass and relative coordinates

$$\vec{R} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2\,, \qquad \vec{r} = \vec{r}_2 - \vec{r}_1\,. \tag{1.59}$$

Conversely we have

$$\vec{r}_1 = \vec{R} - \frac{m_2}{M}\vec{r}, \qquad \vec{r}_2 = \vec{R} + \frac{m_1}{M}\vec{r}. \tag{1.60}$$

☐ The equations of motion in the new coordinates are

$$M\ddot{\vec{R}} = \vec{F}^{(\text{ext})} \tag{1.61}$$

$$\ddot{\vec{r}} = \frac{\vec{F}_2^{(\text{ext})}}{m_2} + \frac{\vec{F}_{21}}{m_2} - \frac{\vec{F}_{12}}{m_1} - \frac{\vec{F}_1^{(\text{ext})}}{m_1}\,. \tag{1.62}$$

The first equation has already been derived in Sec. (1.5.3).
☐ For a closed system ($F_i^{(\text{ext})} = \vec{0}$) the equations for $\vec{R}$ and $\vec{r}$ decouple:

$$\boxed{M\ddot{\vec{R}} = \vec{0} \qquad \mu\ddot{\vec{r}} = \vec{F}_{21}} \tag{1.63}$$

where

$$\mu = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} \tag{1.64}$$

is the reduced mass. In this case the relative motion becomes effectively a one-body problem.

☐ The kinetic energy is additive in the new coordinates, i.e. it decouples into two terms for center of mass and relative motion

$$T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2 \,. \tag{1.65}$$

This is also true for systems that are not closed. Proof: Plug the expressions (1.60) into $m_1\dot{r}_1^2/2 + m_2\dot{r}_2^2/2$. The mixed terms cancel.

☐ The angular momentum is additive in the new coordinates, i.e.

$$\vec{L} = \vec{R} \times \dot{\vec{P}} + \vec{r} \times \dot{\vec{p}} \tag{1.66}$$

where $\vec{p} = \mu\dot{\vec{r}}$ is the momentum associated with relative motion. Proof: Analogous to the case of kinetic energy.

1.5.6 The Virial Theorem

☐ We consider a bound system of n particles, i.e. all motions $\vec{x}_i(t)$ and velocities $\dot{\vec{x}}_i(t)$ are bounded functions. With that we mean that there exists a distance R and a speed V such that $|\vec{x}_i(t)| < R$ and $|\dot{\vec{x}}_i(t)| < V$ for all allowed t and all $i = 1, \ldots, n$.

☐ We define the time-average of a function $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto f(t)$ as

$$\langle f \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^\infty f(t)dt \,. \tag{1.67}$$

☐ The time-average of the time-derivative of a bounded function (i.e. $|f| < R$) vanishes. Proof:

$$\left\langle \frac{df}{dt} \right\rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^\infty \frac{df}{dt} dt = \lim_{T\to\infty} \frac{f(T) - f(0)}{T} = 0 \tag{1.68}$$

☐ We are interested in the time-average of the kinetic energy of the bound system of particles. We can compute it

$$2\langle T \rangle = \left\langle \sum_{i=1}^n \vec{p}_i \cdot \vec{v}_i \right\rangle = \left\langle \frac{d}{dt} \sum_{i=1}^n \vec{p}_i \cdot \vec{x}_i \right\rangle - \left\langle \sum_{i=1}^n \dot{\vec{p}}_i \cdot \vec{x}_i \right\rangle \tag{1.69}$$

The first term of the right most expression vanishes since it is the average of a time-derivative of a bound function. The second term we can obviously transform further with Newton's Second Law. In particular, for conservative forces we obtain the following theorem.

☐ Virial Theorem: For a bound system of n particles with conservative forces we have

$$\boxed{2\langle T \rangle = \left\langle \sum_{i=1}^n \vec{x}_i \cdot \nabla_i U \right\rangle} \,. \tag{1.70}$$

The expression on the right hand side of the equation is called the *virial of the system*. Here U is the potential energy of the system and ∇_i is the gradient with respect to the coordinates of particle i.

□ The most important applications are provided by cases of central forces between particles that scale as a power γ of the distance between particles, i.e.

$$U_{ij}(\vec{x}_i, \vec{x}_j) = \alpha_{ij}|\vec{x}_i - \vec{x}_j|^\gamma \tag{1.71}$$

between particles i, j, $U = 1/2\sum_{i,j} U_{ij}$. In that case the virial is $\gamma\langle U\rangle$ and hence

$$2\langle T\rangle = \gamma\langle U\rangle\,. \tag{1.72}$$

Proof: We have

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{n}\vec{x}_k\cdot\nabla_k\sum_{i,j=1}^{n}\alpha_{ij}|\vec{x}_i-\vec{x}_j|^\gamma &= \frac{1}{2}\sum_{i,j=1}^{n}\alpha_{ij}\sum_{k=1}^{n}(\delta_{ik}-\delta_{jk})\vec{x}_k\cdot(\vec{x}_i-\vec{x}_j)\gamma|\vec{x}_i-\vec{x}_j|^{\gamma-2}\\ &= \frac{1}{2}\sum_{i,j=1}^{n}\alpha_{ij}|\vec{x}_i-\vec{x}_j|^2\gamma|\vec{x}_i-\vec{x}_j|^{\gamma-2} = \gamma U\,.\end{aligned} \tag{1.73}$$

□ Specific examples:

- $\gamma = -1$ (Gravity, Coulomb):

$$2\langle T\rangle = -\langle U\rangle \quad \Rightarrow \quad E = \langle T\rangle + \langle U\rangle = -\langle T\rangle < 0 \tag{1.74}$$

- $\gamma = 2$ (Hooke): $\langle T\rangle = \langle U\rangle$.