Physics 606 (Quantum Mechanics I) — Spring 2017

Midterm Exam

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[1] **Measurement** (15 points)

Let $\mathbf{A}$ be a Hermitian operator in a Hilbert space of countable dimension. ψ_1 and ψ_2 are two properly normalized eigenstates of $\mathbf{A}$ with eigenvalues a_1 and a_2 respectively ($a_1 \neq a_2$). Suppose at a time $t = 0$ the system is in a state

$$\psi = \psi_1 + \frac{i}{2}\psi_2$$

and a measurement of $\mathbf{A}$ is made on the system at that time.

(a) (10) What are the probabilities that the measurement yields the values a_1, a_2, or neither of those two?

(b) (5) Suppose ψ_2 is also an eigenstate of the Hamilton operator of the system, but ψ_1 is not. If the system evolves from state ψ at time $t = 0$ to some later time $t = t_1$, what can you generally say about the probabilities to measure a_1 and a_2 at that later time?

[2] **Translations Of Operators** (15 points)

Consider the operator $\mathbf{T}_{\vec{a}}$, representing spatial translations by a vector $\vec{a}$ on a Hilbert space. Show that translations of the position and momentum operators $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$ give

$$\mathbf{T}_{\vec{a}}\, \vec{\mathbf{r}}\, \mathbf{T}_{\vec{a}}^{\dagger} = \vec{\mathbf{r}} - \vec{a}\mathbf{1}$$
$$\mathbf{T}_{\vec{a}}\, \vec{\mathbf{p}}\, \mathbf{T}_{\vec{a}}^{\dagger} = \vec{\mathbf{p}}$$

where $\mathbf{1}$ is the identity operator.

[3] **Quantum Corrections to Newton's Second Law** (15 points)

Recall that Ehrenfest's Theorem does not imply that the average position and momentum of a particle stictly follows the classical equations of motion. Here we consider a particle of mass m subject to a potential energy $V(\vec{r})$.

(a) (7) Rederive Ehrenfest's Theorem for the average position and momentum operators $\langle\vec{\mathbf{r}}\rangle$, $\langle\vec{\mathbf{p}}\rangle$ (e.g. using the equation of motion for expectation values). What do you get for the acceleration $d^2/dt^2\langle\vec{\mathbf{r}}\rangle$?

(b) (8) Show that for *slowly varying* potentials $V(\vec{r})$ the quantum mechanical result for the acceleration found in (a) can be written as the classical Newton's Second Law for $\langle\vec{\mathbf{r}}\rangle$ plus a correction proportional to the squared width of the wave packet.
Hint: Taylor expansion

[4] **Quantum Trough** (15 points)

Consider a particle in a 2-dimensional infinite trough of length L and a parabolic cross section, i.e.

$$V(x,y) = \frac{1}{2}m\omega^2 x^2 \quad \text{for } 0 < y < L\,,$$
$$V(x,y) = \infty \quad \text{for } y < 0, y > L.$$

Determine the energy eigenvalues and energy eigenstates of the system. You do not have to normalize the eigenstates.

[5] **Complex Potential** (20 points)

Consider a complex potential energy in the time-dependent Schrödinger equation, i.e. $V(\vec{r}) = V'(\vec{r}) - iV''(\vec{r})$ where both V' and V'' are real. Using the usual ansatz $\psi(\vec{r},t) = A(\vec{r},t)e^{\frac{i}{\hbar}S(\vec{r},t)}$ in the Schrödinger equation with real-valued amplitude A and phase S *derive* the modified Hamilton-Jacobi equation and continuity equation for S and the particle density $\rho = A^2$ in the limit $\hbar \to 0$ in this case.

[6] **Well With One Infinite Wall** (20 points) Consider particles of mass m in 1-D, incident onto two subsequent potential steps. The first one is a step of height $V_0 > 0$ and length L, the second step of infinite height. To be more precise the potential energy function is

$$V(x) = 0 \quad \text{for } x > L\,,$$
$$V(x) = V_0 \quad \text{for } 0 < x < L\,,$$
$$V(x) = \infty \quad \text{for } x < 0\,.$$

Suppose the particles have energy $E > V_0$.

(a) (12) Write down an ansatz for the energy eigen functions everywhere and give all equations from matching/boundary conditions you can find for unknown coefficients in your wave functions. *You don't have to solve these equations.*

(b) (8) Calculate the current j of the Schrödinger field in all 3 parts of the system ($x < 0, 0 < x < L, x > L$) using the wave functions from (a). Show that the energy eigenfunctions are standing waves.

Useful Formulae

- δ-function

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{ik(x-x_0)}dk = \delta(x-x_0) \tag{1}$$

- Hamilton-Jacobi for the classical action $S(\vec{r},\vec{p},t)$

$$\frac{\partial S}{\partial t} + H(\vec{r},\vec{p}) = 0 \quad \text{with } p_i = \frac{\partial S}{\partial r_i} \tag{2}$$

- Current of the Schrödinger field

$$\vec{j}(\vec{r},t) = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \tag{3}$$

- Jacobi identity

$$[F,[G,H]] + [H,[F,G]] + [G,[H,F]] = 0 \tag{4}$$

- Baker Campbell Hausdorff (if A, B commute with their commutator!)

$$e^A e^B = e^{A+B+[A,B]/2} \tag{5}$$

- Virial theorem for *stationary* states

$$2\langle T\rangle = \langle \vec{r}\cdot\nabla V\rangle \tag{6}$$

- Closure/completeness for continuous spectrum with eigenstates ψ_α

$$\int_{\text{spec}} \psi_\alpha^*(\vec{r}')\psi_\alpha(\vec{r})d\alpha = \delta^{(3)}(\vec{r}'-\vec{r}) \tag{7}$$

- Generator of Galilei boosts

$$\vec{K} = m\vec{r} - \vec{p}t \tag{8}$$

- Hermite polynomials

$$\frac{d^2}{d\xi^2}H_n(\xi) - 2\xi\frac{d}{d\xi}H_n(\xi) + 2nH_n(\xi) = 0 \tag{9}$$

$$\frac{d}{d\xi}H_n(\xi) = 2nH_{n-1}(\xi) \tag{10}$$

$$F(\xi,s) = \sum_{n\in\mathbb{N}} H_n(\xi)\frac{s^n}{n!} = e^{\xi^2-(s-\xi)^2} \tag{11}$$

- Harmonic oscillator: orthonormal energy eigenstates

$$\psi_n(x) = 2^{-\frac{n}{2}} n!^{-\frac{1}{2}} \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2} \tag{12}$$

- A useful integral

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}dx = \frac{1}{2}\sqrt{\pi} \tag{13}$$