Physics 606 — Spring 2017

Homework 8

Instructor: Rainer J. Fries *Turn in your work by April 13*

[1] **Examples of Scattering and Bound States in 1-D** (30 points)

For the numerical and graphing work here you can use a program or programming language of your choice.

(a) Consider electrons of energy E interacting with a square potential well of depth 1 eV and width 1 nm. How many bound states are there in the potential well? Determine their energy eigenvalues graphically or numerically as well as you can. Write down the corresponding eigenstate wave functions *inside* the well as a superposition of $\sin$ and $\cos$ functions.

(b) Using the results from the lecture, plot the transmission coefficient T and the phase shift (for transmission) as a function of energy E if a beam of electrons with kinetic energy $E > 0$ is interacting with the well. Plot or sketch the result in the range $0 < E < 20$ eV.

(c) Carry out the same study as in (b) but for an electron beam interacting with a potential *barrier* of 1 eV height and a width of 1 nm. Plot the results again in the range $0 < E < 20$ eV.

[2] **Triangular Potential – Exact Solution** (20 points)

Consider a particle of mass m in a linear confining potential $V(x) = b|x|$.

(a) Show that the time-independent Schrödinger equation in this case can be rewritten as a differential equation of the type

$$\frac{d^2}{dx^2}\psi - x\psi = 0\,. \tag{1}$$

The solutions to this equation are the famous Airy-functions $Ai(x)$ and $Bi(x)$ with $\lim_{x\to\infty} Ai(x) = 0$ and $\lim_{x\to\infty} Bi(x) = \infty$. If you are not familiar with Airy functions you can find basic information at
`http://mathworld.wolfram.com/AiryFunctions.html`

(b) Now you can discuss the energy eigenfunctions and eigenvalues for this potential. Give the two lowest energy eigenvalues explicitly (the zeros of Ai and its derivative Ai' with smallest absolute values are -2.33811 and -1.01879, respectively).

[3] **Triangular Potential in 1-Parameter Approximations** (20 points)

Consider again the situation of problem [2].

(a) Approximate the ground state solution by a Gaussian function of type $e^{-\alpha^2 x^2}$ with parameter α. Find the value of α that makes the functional $\langle H \rangle$ stationary. Compare the energy eigenvalue you obtain for the ground state with the true value from [3].

(b) Repeat the discussion using a Gaussian with one node of type $xe^{-\alpha^2 x^2}$ as an approximation for the first excited state. Again determine the best value for the energy eigenvalue and compare to the result of [3].

(c) Repeat (a) by using the function $x^2 e^{-\alpha x^2}$. Is the trial function in (a) or (b) better suited to approximate the ground state energy? Would the trial function in (c) be a good approximation for the second excited state?

[4] **Angular Momentum Operators** (30 points)

(a) Show the following commutation relations for the angular momentum operator $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$:

(i) $[\mathbf{L}_j, \mathbf{L}_k] = i\hbar\epsilon_{jkl}\mathbf{L}_l$, $j, k, l = 1, 2, 3$ where ϵ_{jkl} is the usual anti-symmetric Levi-Civita tensor with $\epsilon_{123} = 1$;

(ii) $[\mathbf{L}_j, \mathbf{L}^2] = 0$ for $j = 1, 2, 3$ where $\mathbf{L}^2 = \mathbf{L}_1^2 + \mathbf{L}_2^2 + \mathbf{L}_3^2$.

(b) Derive the nabla operator ∇ and the Laplace operator $\triangle$ in spherical coordinates r, θ, ϕ.

(c) Give explicit expressions of the operators $\mathbf{L}_x$, $\mathbf{L}_y$ and $\mathbf{L}_z$, in coordinate space represenations in *spherical coordinates* and show that in particular

$$\mathbf{L}^2 = -\hbar^2 \left[\frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) \right] . \tag{2}$$