Physics 606 — Spring 2017

Homework 7

Instructor: Rainer J. Fries *Turn in your work by April 4*

[1] **Scattering off a 1-D Square Potential** (35 points)

(a) Consider a potential barrier of height V_0 and width $2a$ as introduced in III.2.2 in class. Discuss energy eigenstates with energy above the barrier height, i.e. $E > V_0$. What is the general form of the energy eigenfunctions? Derive the M-matrix from the matching conditions and discuss the transmission and reflection coefficients T and R.

(b) Repeat the discussion for a potential well of depth $-V_0$ and width $2a$ as in III.3 in class. Discuss unbound energy eigenstates, i.e. $E > 0$. What is the general form of the energy eigenfunctions? Derive the M-matrix from the matching conditions and discuss the transmission and reflection coefficients T and R.
Hint: If you can find similarities between the situations in (a) and (b) part (b) will be very little work.

[2] **Properties of Wigner Distributions** (30 Points)

Consider the Wigner distribution $W(\vec{r}, \vec{p}, t)$ associated with a wave function $\psi(\vec{r}, t)$ as defined in class.

(a) Show that $W(\vec{r}, \vec{p}, t)$ is a real-valued function.

(b) Prove that universal upper and lower bounds for W exist, more precisely

$$-\frac{2}{h^3} \leq W(\vec{r}, \vec{p}, t) \leq \frac{2}{h^3} \,. \tag{1}$$

(c) Consider a second Wigner function $W'(\vec{r}, \vec{p}, t)$ which is associated with a different wave function $\psi'(\vec{r}, t)$. Show that the overlap probability of the wave functions is given by

$$|\langle \psi' | \psi \rangle|^2 = (2\pi\hbar)^3 \int W(\vec{r}, \vec{p}, t) W'(\vec{r}, \vec{p}, t) d^3r d^3p \,. \tag{2}$$

[3] **Hamilton's Principle for Fields** (35 points)

Consider a field $\psi(x)$ as a function of coordinates $x = (x_i)_{i=1}^N$. Let $\mathcal{L}(\psi, \frac{\partial \psi}{\partial x_j}, x)$ be the *Lagrange density* for ψ, depending on ψ, its first derivatives, and the position vector x. Let

$$S[\psi] = \int_\Gamma \mathcal{L}\left(\psi, \frac{\partial \psi}{\partial x_j}, x\right) dx^N \tag{3}$$

be the action defined as an integral of the Lagrange density over a region Γ in $\mathbb{R}^N$. In the following we only consider fields ψ that take fixed values on the boundary of Γ, denoted as $\partial\Gamma$. Show that the following two statements are equivalent:[1]

[1]This statement can be easily generalized to a Lagrange density involving several fields $\psi_i(x)$, as for example required for the complex Schrödinger field.

(i) $\psi(x)$ is an extremum of the functional S, i.e. small variations $\delta\psi(x)$ around $\psi(x)$ consistent with the boundary conditions leave S invariant: $\delta S = 0$.

(ii) ψ satiesfies the Euler-Lagrange field equation

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x_j} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi}{\partial x_j} \right)} = 0. \tag{4}$$

Hint: (1) The standard derivation parameterizes small deviations from $\psi(x)$ as $\psi(x, \alpha) = \psi(x) + \alpha\eta(x)$ where α is a "small" parameter and $\eta(x)$ is a test function which has to vanish on $\partial\Gamma$ but is otherwise arbitary. Then $\delta S = (\partial S/\partial\alpha)\delta\alpha$; OR (2) take your favorite classical mechanics textbook, look up the derivation of the Euler-Lagrange equations from the Hamilton Principle when ψ is only a function of one parameter (time in classical mechanics!) and generalize it to the case of a multi-dimensional parameter space.