
Physics 606 — Spring 2017

Homework 6

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Turn in your work by March 21

[1] Hermite Polynomials (30 points)

(a) Show that for Hermite polynomials of degree $n \in \mathbb{N}$

$$\frac{d}{d\xi} H_n(\xi) = 2n H_{n-1}(\xi) \quad (1)$$

Hint: Use the power series for Hermite Polynomials

(b) Prove that the generating function

$$F(\xi, s) = \sum_{n=0}^{\infty} \frac{H_n(\xi)}{n!} s^n \quad (2)$$

can be written in closed form as

$$F(\xi, s) = e^{\xi^2 - (s-\xi)^2} \quad (3)$$

Hint: Derive a simple differential equation for F which follows from the relation in (a).

(c) Using the generating function show that

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{dx^n} e^{-\xi^2} \quad (4)$$

(d) Prove the integral representation

$$H_n(\xi) = \frac{2^n}{\sqrt{\pi}} \int_{\mathbb{R}} (\xi + is)^n e^{-s^2} ds. \quad (5)$$

Hint: Induction using (a).

[2] Important Amplitudes for the Harmonic Oscillator (35 points)

(a) Consider the function

$$I(s, t, \lambda) = \int_{\mathbb{R}} F(\xi, s) F(\xi, t) e^{-\xi^2 + 2\lambda\xi} d\xi \quad (6)$$

which contains the generating function F for Hermite polynomials. Show with the help of problem [1] that

$$I(s, t, \lambda) = \sqrt{\pi} e^{\lambda^2 + 2(st + \lambda s + \lambda t)}, \quad (7)$$

and on the other hand

$$I_{nmk} := \int_{\mathbb{R}} H_n(\xi) H_m(\xi) \xi^k e^{-\xi^2} d\xi = \frac{1}{2^k} \frac{\partial^{n+m+k} I}{\partial s^n \partial t^m \partial \lambda^k} \Big|_{s,t,\lambda=0} \quad (8)$$

for $n, m, k \in \mathbb{N}$. Thus I is a generating function for integrals of the type I_{nmk} .

Hint: We have discussed the special case $k = 0$ in class.

- (b) Consider a particle of mass m in a harmonic oscillator potential $\frac{1}{2}m\omega^2x^2$. Use the results from (a) to prove that for stationary states $\psi_n(x)$

$$\langle \psi_n | x | \psi_{n'} \rangle = \int_{\mathbb{R}} \psi_n(x) x \psi_{n'}(x) dx = \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{n} \delta_{n',n-1} + \sqrt{n+1} \delta_{n',n+1} \right]. \quad (9)$$

for $n, n' \in \mathbb{N}$

- (c) In the same situation as in (b), what is $\langle \psi_n | x^2 | \psi_{n'} \rangle$?

[3] **Harmonic Oscillator: Averages and Virial Theorem** (35 points)

Consider a particle of mass m in a harmonic oscillator potential $\frac{1}{2}m\omega^2x^2$.

- (a) Use the *explicit solution* for the time evolution of the wave function $\psi(x, t)$ with given initial condition $\psi(x, 0)$ at $t = 0$ to show that the expectation values of position and momentum of the particle follow a classical motion, i.e.

$$\langle x \rangle(t) = \langle x \rangle(0) \cos \omega t + \frac{\langle p \rangle(0)}{m\omega} \sin \omega t \quad \langle p \rangle(t) = m \frac{d\langle x \rangle(t)}{dt} \quad (10)$$

Hint: Write ψ as a superposition of energy eigenstates and use the results of problem [2].

- (b) Calculate $\langle \psi_n | p^2 | \psi_{n'} \rangle$, $n, n' \in \mathbb{N}$ for any stationary states ψ_n (p is the momentum operator). Use the result to validate the virial theorem for stationary states.