Physics 606 — Spring 2017

Homework 4

Instructor: Rainer J. Fries — *Turn in your work by March 2*

[1] **Baker-Campbell-Hausdorff Relation** (25 points)

Let F, G be two operators on a vector space of functions. Prove the following simplified Baker-Campbell-Hausdorff formula if F and G both commute with their commutator $[F, G]$:

$$e^F e^G = e^{F+G+[F,G]/2} \,. \tag{1}$$

Hint: Consider the expression $d(e^{tF}e^{tG})/dt$ where t is a real-valued parameter. Establish a differential equation for $e^{tF}e^{tG}$ which you can easily solve.

[2] **Commutators and Poisson Brackets; Products of Conjugate Variables** (25 points)

Let $H = p^2/2m + V(\vec{r})$ be the Hamilton operator of a system.

(a) Calculate the commutator $[\vec{r}\cdot\vec{p}, H]$ where $\vec{r}$ and $\vec{p}$ are the position and momentum operator, respectively. Show that the commutator $[\vec{p}\cdot\vec{r}, H]$ results in the same operator.

(b) Compute $[(\vec{r}\cdot\vec{p})(\vec{r}\cdot\vec{p}), H]$ and $[(\vec{r}\cdot\vec{p})(\vec{p}\cdot\vec{r}), H]$ for the special case $V = 0$. Compare both resulting operators by applying them to a plane wave test function $e^{i(\vec{r}\cdot\vec{p}-Et)/\hbar}$.

(c) Calculate the classical Poisson bracket[1] $\{\vec{r}\cdot\vec{p}, H\}$ and compare to the result of (a). Also compare the fundamental commutators and Poisson brackets $[r_i, p_j]$ and $\{r_i, p_j\}$, $i, j = 1, 2, 3$. Does the correspondence principle between commutators of operators and Poisson brackets of their classical counterparts work in those cases?

(d) For the special case $V = 0$ calculate the classical Poisson bracket $\{(\vec{r}\cdot\vec{p})^2, H\}$ and compare to the results of (b). Is the correspondence principle tenable?

[3] **Schrödinger Equation with Electromagnetic Potential** (30 points)

Recall that the classical Hamilton function for a particle of mass m and charge q subject to electrostatic and magnetic potentials $\phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$, respectively, is

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi \,. \tag{2}$$

(a) What are the Hamilton-Jacobi equation for the classical action S_{cl} and the classical continuity equation in this case?
Hint: Note that in the presence of a vector potential $\vec{A}$ the relevant velocity is $(\vec{p} - q\vec{A})/m$.

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[1] We agree to use the definition $\{f, g\} = \sum_{k=1}^{n}\left(\frac{\partial f}{\partial r_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial r_k}\right)$.

(b) Show that the *Schrödinger equation with electromagnetic potentials*

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2\psi(\vec{r},t) + q\phi\psi(\vec{r},t) \tag{3}$$

for a wave function

$$\psi = C(\vec{r},t)e^{\frac{i}{\hbar}S(\vec{r},t)} \tag{4}$$

with real amplitude C and phase S reduces to the two classical equations from (a) for $\hbar \to 0$ and $S \to S_{\mathrm{cl}}$.

[4] **Gauge Invariance** (20 points)

Show that the Schrödinger equation from problem [3](b) is invariant under the simultaneous gauge transformations

$$\vec{A} \mapsto \vec{A} + \nabla f \qquad \phi \mapsto \phi - \frac{\partial f}{\partial t} \qquad \psi \mapsto e^{\frac{i}{\hbar}qf}\psi \tag{5}$$

where $f(\vec{r},t)$ is a real-valued function.