Physics 606 — Spring 2017

Homework 3

Instructor: Rainer J. Fries *Turn in your work by February 16*

[1] **Operator Algebra** (25 points)

(a) Let F and G be two operators on a vector space of functions. They both commute with their commutator $[F, G]$. Show that for any $n \in \mathbb{N}$

$$[F, G^n] = nG^{n-1}[F, G] \tag{1}$$

$$[F^n, G] = nF^{n-1}[F, G] \tag{2}$$

(b) Prove that the Jacobi identity

$$[F, [G, H]] + [H, [F, G]] + [G, [H, F]] = 0 \tag{3}$$

holds for arbitrary operators F, G, H.

[2] **Commutators in Coordinate and Momentum Space** (25 points)

Let $F(\vec{r})$ and $G(\vec{p})$ be two physical quantities as a function of coordinate $\vec{r}$ and momentum $\vec{p}$ respectively. Let F_p, G_p and F_r, G_r be the operators representing F and G in momentum and coordinate space respectively, acting on spaces of sufficiently fast falling functions $\mathcal{S}_r$ and $\mathcal{S}_p$ respectively. Prove that the commutators of F and G in both representations are related by Fourier transformation, i.e.

$$[F_r, G_r] f(\vec{r}) = (2\pi\hbar)^{-3/2} \int_{\mathbb{R}^3} [F_p, G_p] \hat{f}(\vec{p}) e^{\frac{i}{\hbar}\vec{r}\cdot\vec{p}} d^3p \tag{4}$$

for any test function $f \in \mathcal{S}_r$ with Fourier transform $\hat{f} \in \mathcal{S}_p$.

Note: This can easily be generalized to a proof of the statement in II.3.2 in the lecture which makes the same statement for arbitary F, G as long as pairs of conjugate variables are separable.

[3] **Time Evolution of Wave Packet Widths** (25 points)

Using the equation of motion for expectation values show that for a free particle of mass m in one dimension the usual variances $(\Delta p)^2 = \langle (p - \langle p \rangle)^2 \rangle$, $(\Delta x)^2 = \langle (x - \langle x \rangle)^2 \rangle$ in momentum and coordinate space have the time dependences

$$(\Delta p)^2(t) = (\Delta p)^2(0) = \text{const.}, \tag{5}$$

$$(\Delta x)^2(t) = (\Delta x)^2(0) + \frac{2}{m}\left[\frac{1}{2}\langle xp + px \rangle (0) - \langle x \rangle(0)\langle p \rangle(0)\right] t + \frac{(\Delta p)^2(0)}{m^2} t^2 , \tag{6}$$

if they exist.

[4] **Gaussian Wave Packets — Part III** (25 points)

(a) Use the result of [3] to calculate $(\Delta x)^2$ and $(\Delta p)^2$ as function of time t for the propagating Gaussian wave packet from HW 2, problem [1].

(b) Now calculate $(\Delta x)^2$ and $(\Delta p)^2$ directly from the explicit results of HW 2, problem [1] and compare to (a).