Physics 606 — Spring 2017

Homework 2

Instructor: Rainer J. Fries *Turn in your work by February 2*

[1] **Gaussian Wave Packets — Part II** (25 points)

At time $t = 0$ consider a Gaussian wave packet (cf. HW I, [2]) centered around x_0 with an average momentum k_0, i.e.

$$\psi(x, 0) = \left(\sqrt{2\pi}\sigma\right)^{-1/2} e^{ik_0 x} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \tag{1}$$

(a) If this packet represents a free particle of energy $E = \hbar k_0^2/2m$ calculate the time evolution $\psi(x, t)$ in explicit form (i.e. all integrals should be carried out in your final result).

(b) Using a program of your choice (Mathematica, Matlab, etc.) plot the real and imaginary parts of $\psi(x, t)$ as well as $|\psi(x, t)|^2$ as functions of x for different values of t. Choose suitable parameters to document the spreading of the wave packet with time. What determines the "speed" with which the width of the wave packet increases?

[2] **Schrödinger Equation in Momentum Space** (20 points)

Starting from the Schrödinger equation in coordinate space for a particle of mass m and potential energy $V(\vec{r})$ explicitly show that

$$i\hbar\frac{\partial}{\partial t}\phi(\vec{p}, t) = \frac{p^2}{2m}\phi(\vec{p}, t) + V(i\hbar\nabla_p)\phi(\vec{p}, t) \tag{2}$$

where $\phi(\vec{p}, t)$ is the Fourier transformation of a coordinate space solution $\psi(\vec{r}, t)$, if $V(\vec{r})$ has a representation as a power series.

[3] **Continuity Equation** (30 points)

(a) Consider a system of classical particles described by a density $\rho(\vec{r}, t)$ and a velocity field $\vec{v}(\vec{r}, t)$. Consider a volume $V \equiv V(t)$ that is co-moving with the particle flow, such that the particle number in this volume is conserved,

$$\frac{d}{dt}\int_{V(t)} \rho\, d^3r = 0\,. \tag{3}$$

From this condition derive the continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho\vec{v}) = 0\,. \tag{4}$$

Hint: You can consider a very small volume ΔV and expand the total derivative with respect to time in terms of partial derivatives.

(b) Consider wave functions $\psi_1(\vec{r}, t)$ and $\psi_2(\vec{r}, t)$ which obey the same time-dependent Schrödinger equation. Derive the current density $\vec{j}_{12}(\vec{r}, t)$ which fulfills the continuity equation

$$\frac{\partial}{\partial t}(\psi_1\psi_2^*) + \nabla \cdot \vec{j}_{12} = 0\,. \tag{5}$$

(c) Hence what is the current $\vec{j}$ that satisfies the continuity equation for the probability density $\rho = |\psi|^2$ of a single field ψ? Show that $\vec{j}$ goes towards the result from (a), i.e. $\vec{j} \to \rho\vec{v}$ in the classical limit.

[4] **Discontinuous Potential Energy** (25 points)

Consider a potential energy function $V(\vec{r})$ which is discontinuous at the plane $z = 0$. Show that despite the discontinuity the wave function ψ and its derivative have to be continuous at the plane $z = 0$, i.e.

$$\lim_{\epsilon\to 0} \psi(x, y, -\epsilon, t) = \lim_{\epsilon\to 0} \psi(x, y, +\epsilon, t) \tag{6}$$

$$\lim_{\epsilon\to 0} \frac{\partial}{\partial z}\psi(x, y, -\epsilon, t) = \lim_{\epsilon\to 0} \frac{\partial}{\partial z}\psi(x, y, +\epsilon, t) \tag{7}$$

for each point $(x, y, 0)$ on the surface and for all times t.
Hint: First consider the continuity of the probability current $\vec{j}$ by using a suitably chosen test volume that contains a piece of the surface.