
Physics 606 — Spring 2017

Homework 10

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Turn in your work by April 27

[1] **Some More Commutators of Angular Momentum Operators** (20 points)

- (a) Compute the commutators $[\mathbf{L}_i, \mathbf{r}_j]$, $[\mathbf{L}_i, \mathbf{p}_j]$ and $[\mathbf{L}_i, \mathbf{K}_j]$, $i = 1, 2, 3$, where \mathbf{L}_i , \mathbf{r}_i , \mathbf{p}_i and \mathbf{K}_i are the orbital angular momentum, position, momentum, and boost operators, respectively.
- (b) For the raising and lowering operators $\mathbf{L}_\pm = \mathbf{L}_x \pm i\mathbf{L}_y$ compute the commutators $[\mathbf{L}_\pm, \mathbf{L}_z]$ and $[\mathbf{L}_+, \mathbf{L}_-]$. Show that $\mathbf{L}^2 - \mathbf{L}_z^2 = \mathbf{L}_+ \mathbf{L}_- \mp \hbar \mathbf{L}_z$

[2] **Harmonic Oscillator Algebra** (40 points)

- (a) Calculate the matrix representation of the lowering and raising operators \mathbf{a} and \mathbf{a}^\dagger of the harmonic oscillator with respect to the energy eigenstate basis $|n\rangle$, $n \in \mathbb{N}$. I.e. calculate the matrix elements $\langle n' | \mathbf{a} | n \rangle$, etc.
- (b) With the help of (a) determine the normalization factors C_n, D_n, N_n in the following equations from lecture:

$$\mathbf{a}|n\rangle = C_n|n-1\rangle \quad \mathbf{a}^\dagger|n\rangle = D_n|n+1\rangle \quad |n\rangle = N_n(\mathbf{a}^\dagger)^n|0\rangle \quad (1)$$

- (c) Compute the matrix representation of the position operator \mathbf{x} with respect to the basis $|n\rangle$, $n \in \mathbb{N}$.
- (d) Consider the trivial eigenvalue equation $\mathbf{x}|x\rangle = x|x\rangle$ where \mathbf{x} is the position operator, and x the position described by the eigenstate $|x\rangle$. Derive the corresponding “matrix equation” that is the eigenvalue equation for the amplitudes $\psi_n(x) = \langle n|x\rangle$ of the positions in the basis $|n\rangle$, $n \in \mathbb{N}$.

Hint: Insert a complete set of states.

- (e) Of course the $\psi_n(x)$ are just the complex conjugates of the coordinate space wave functions of the harmonic oscillator, $\langle x|n\rangle$. Show that the eigenvalue problem for the $\psi_n(x)$ in (d) is solved by

$$\psi_n(x) = 2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}h_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\psi_0(x) \quad (2)$$

where the $h_n(x)$ satisfy the recurrence relation

$$h_{n+1}(x) - 2xh_n(x) + 2nh_{n-1}(x) = 0. \quad (3)$$

- (f) Show that the Hermite polynomials satisfy the recurrence relations from (e), so that $h_n(x) = CH_n(x)$.

Hint: You can, e.g., use the relation you proved in HW VI, [1](c).

Note: You could get the remaining unknowns $C\psi_0(x)$ which contain the Gaussian and the normalization factor from the closure relation, but you don't need to do that here.

[3] **Free Particle Solutions in Spherical Coordinates** (40 points)

Consider a free particle of mass m . Since \mathbf{L}^2 and \mathbf{L}_z commute with the free Hamilton operator \mathbf{H} , you can find simultaneous eigenfunctions $\psi(\vec{r}) = \psi(r, \theta, \phi)$ for \mathbf{H} , \mathbf{L}^2 and \mathbf{L}_z .

- (a) The ansatz $\psi_{l,m}(r, \theta, \phi) = CR(r)Y_l^m(\theta, \phi)$, where the $Y_l^m(\theta, \phi)$ are the spherical harmonics from HW IX, [2], obviously provides eigenfunctions of \mathbf{L}^2 and \mathbf{L}_z with eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$. Show that the “radial” differential equation that $R(r)$ needs to satisfy to make $\psi_{l,m}$ an eigenfunction of H as well is

$$\frac{d^2 R}{d\rho^2} + \frac{2}{\rho} \frac{dR}{d\rho} + \left[1 - \frac{l(l+1)}{\rho^2} \right] R = 0 \quad (4)$$

where we have introduced the dimensionless radial coordinate $\rho = r\sqrt{2mE}/\hbar$.

- (b) The radial equation above is also called the spherical Bessel equation. Its regular solutions are the spherical Bessel functions j_l which are in integral representation defined as

$$j_l(\rho) = \frac{\rho^l}{2^{l+1}l!} \int_{-1}^1 e^{i\rho s} (1-s^2)^l ds. \quad (5)$$

(We will not be interested in its singular solutions which are the spherical Neumann functions n_l). Show that the j_l given above satisfy the radial equation from (a).

- (c) Show that

$$j_l(z) = (-1)^l \rho^l \left(\frac{d}{\rho d\rho} \right)^l \frac{\sin \rho}{\rho}. \quad (6)$$

Give the first 3 spherical Bessel functions ($l = 0, 1, 2$) explicitly and graph them.

- (d) We already know that plane waves are eigenfunctions of \mathbf{H} that are also simultaneously eigenfunctions for the components of the momentum vector \vec{p} . Assume a particle in a stationary plane wave $e^{i\vec{k}\cdot\vec{r}}$ with wave vector $\vec{k} = k\hat{e}_z$ pointing in z -direction. Derive an expression that describes this plane wave in terms of angular momentum eigenstates.