Chapter 2

Lagrange Mechanics

We start this chapter with a very brief study of variational methods that are needed to appreciate Lagrangian methods.

2.1 Variational Calculus

☐ We consider the following problem: Let $y :\longrightarrow \mathbb{R}$, $x \rightarrow y(x)$ with $I = [x_1, x_2] \subset \mathbb{R}$ be a differentiable function. For each functional $\Phi : C^1(I) \longrightarrow \mathbb{R}$ and real numbers y_1, y_2 we ask which such function fulfills the following two conditions.[1]

(i) $y(x_1) = y_1$, $y(x_2) = y_2$ (fixed values at the boundaries)

(ii) The given functional Φ has an extremum for $y(x)$.

☐ Here we are interested in functionals of a very specific form, integrals over functions that depend on y, the first derivative $y' = dy/dx$ and x. More specifically, let

$$\Phi[y] = \int_{x_1}^{x_2} dx f\left(y(x), y'(x), x\right) \tag{2.1}$$

where $f : \mathbb{R} \times \mathbb{R} \times I \longrightarrow \mathbb{R}$ is a differentiable function in three variables.
☐ We want to convert the variational problem into the more familiar problem of solving a differential equation. For that parameterize the potential solutions. Let $y(x)$ be the true solution and $y(x, \alpha) = y(x) + \alpha\eta(x)$ a set of other functions conntected smoothly with the solutions. For the test function η let $\eta(x_1) = 0 = \eta(x_2)$ such that $y(x, \alpha)$ satisfies condition (i) for all α.

For (ii) to be fulfilled it is a necessary condition that the variation of Φ with respect to α vanishes. It is furthermore a satisfactory condition for (ii) if such variations vanish for all possible choices of test functions $\eta(x)$.
☐ The variation if α is changed by a small amount $\delta\alpha$ is

$$\delta\Phi = \frac{d\Phi\left[y(x, \alpha)\right]}{d\alpha}\delta\alpha = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y}\frac{dy}{d\alpha} + \frac{\partial f}{\partial y'}\frac{dy'}{d\alpha}\right)\delta\alpha\,. \tag{2.2}$$

We note that $dy'/d\alpha = d/dx(dy/d\alpha)$ since α is independent of x and we can apply partial integration to the second term

$$\int_{x_1}^{x_2} dx \frac{\partial f}{\partial y'}\frac{d}{dx}\frac{dy}{d\alpha} = \left[\frac{\partial f}{\partial y'}\frac{dy}{d\alpha}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \frac{dy}{d\alpha}\frac{d}{dx}\frac{\partial f}{\partial y'}\,. \tag{2.3}$$

[1] $C^1(I)$ denotes the space of differentiable real functions on I.

Note that the variation $dy/d\alpha$ vanishes at the boundaries thus the boundary term vanishes and we arrive at

$$\delta\Phi = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \frac{dy}{d\alpha}\delta\alpha \tag{2.4}$$

We can interpret the term in parentheses as the variation of f with respect to y, $\delta f/\delta y$, and we can identify infinitesimal variations of the curve $y(x)$ with $\delta y = (dy/d\alpha)\,\delta\alpha$ where we let the test function η run to explore all possible variations.

□ Obviously we define an extremum as the condition that $\delta\Phi = 0$ for all possible variations δy. We conclude that

$$\delta\Phi = 0 \text{ for all } \delta y \Longleftrightarrow \frac{\delta f}{\delta y} = 0 \Longleftrightarrow \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0} \tag{2.5}$$

The equation of the right hand side is called the *Euler-Lagrange Equation* for Φ. In particular we have now rephrased the variational problem as the solution to a differential equation: $y(x)$ is an extremum of the functional Φ if and only if it satisfies the Euler-Lagrange equation.

□ Example: Find the shortest path between points (x_1, y_1) and (x_2, y_2) in the x-y-plane.

We can assume that $\vec{r}(x) = (x, y(x))$ is a valid parameterization of the trajectory.[2] The length of the trajectory $s[y]$ is a functional of possible functions $y(x)$ and we look to minimize s. The length is given by

$$s = \int_{x_1}^{x_2} dx \left| \frac{d\vec{r}}{dx} \right| = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx \tag{2.6}$$

Obviously $s[y]$ of the type discussed above with $f(y') = \sqrt{1 + y'^2}$. The Euler-Lagrange Equation gives

$$0 = \frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} \tag{2.7}$$

Hence $y'/\sqrt{1 + y'2} = \text{const.}$ and thus $y' = \text{const.}$ That means that $y = ax + b$ is a straight line as expected. The parameters a and b are determined by the boundary points.

2.2 Constraints and Generalized Coordinates

□ Constraints in mechanical systems are conditions imposed on a motion, e.g. by geometry. They should be included in the equations of motion but this is often difficult to do from the outset. As an example consider a two-particle system with a fixed distance (like a dumbbell with a handle of negligible mass).

□ Constraints are enforced by forces of constraint, e.g. normal forces (solid surfaces), tension (ropes) etc. We note that those forces are a priori not known but they are determined by the motion of the system. In other words only the outcome of the forces (the constraint) not the force itself is known. This makes constraints often challenging to treat.

□ Classification of Constraints. Let us assume a system described by cartesian coordinates $x_1, \ldots, x_N$.

- *Holonomic* constraints are of the form

$$f_i(x_1, \ldots, x_N, t) = 0 \quad \text{for } i = 1, \ldots, k \tag{2.8}$$

[2] Note that we have immediately made the assumption that the trajectory can be characterized as a function $y(x)$ (or $x(y)$). One can reiterate the argument in this example to show that this is always the case.

with $k < N$, where all equations $f_1, \ldots, f_k$ are independent of each other (i.e. the rank of the matrix $\partial f_i/\partial x_j$ is k). Holonomic constraints decrease the effective number of degrees of freedom by k.
Example: Consider the "dumbbell" from above. The constraint can be written as $(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-d^2=0$ where d is the distance of the two point masses and (x_i, y_i, z_i) is the position of mass i. The effective number of degrees of freedom is $N-k=6-1=5$. Later we can identify them as the 3 degrees of freedom of the center of mass point and two rotational degrees of freedom.

- *Non-holonomic* constraints can not be written in the form above. While many types of non-holonomic constraints exist they all have in common that they do not allow for the elimination of a coordiante, i.e. they do not reduce the number of degrees of freedom. The most important examples are the following:

  – Non-integrable constraints of the form

$$\sum_{j=1}^{N} \omega_j^i dx_j + \omega_0^i dt = 0 \quad \text{for } i = 1, \ldots, k\,. \tag{2.9}$$

  Obviously holonomic conditions always imply such (weaker) relations as well. However non-integrable specifically means that there are *no* functions $f_1, \ldots, f_k$ such that $\omega_j^i = \partial f_i/\partial x_j$ and $\omega_0^i = \partial f_i/\partial t$.

  – Inequalities. Examples are constraints by solid surfaces like a particle on a table (constraint $z > 0$ in obvious coordinate system) or a particle sliding off a sphere with radius R ($x^2+y^2+z^2 > R^2$ in coordinate system centered in the sphere).

□ An example for non-integrable constraints will be discussed in HW III.
□ In principle we can deal with constraints by taking into account forces of constraint in Newton's Second Law. E.g. for a single particle we have

$$m\ddot{\vec{x}} = \sum_{j=1}^{l} \vec{F}_j + \sum_{i=1}^{k} \vec{Z}_i \tag{2.10}$$

where the $\vec{Z}_i$ are forces of constraint and the $\vec{F}_j$ are all other forces. Usually the $\vec{Z}_i$ are not known a priori and have to be solved for or eliminated from the equations of motion.
□ In order to eliminate k holonomic constraints we can go from cartesian coordinates $x_1, \ldots, x_N$ to generalized coordinates $q_1, \ldots, q_s$ where $s = N - k$. The following obvious condition needs to be fulfilled for generalized coordinates to be useful: The old coordinates are uniquely determined by the generalized coordinates plus constraints, i.e.

$$x_i = x_i(q_1, \ldots, q_s) \quad \text{for } i = 1, \ldots, N\,. \tag{2.11}$$

If in addition we want to eliminate *all* holonomic constraints we demand that there are no functions f for which $f(q_1, \ldots, q_s) = 0$. Note that we do not demand that generalized coordinates have the dimension of length. They could be e,.g., angles, areas, or any other type of physical quantity. The time derivatives $\dot{q}_1, \ldots, \dot{q}_s$ are called the generalized velocities.

☐ Of course, in the case discussed in the last paragraph the s generalized coordinates are also uniquely determined by the set $x_1, \ldots x_N$ through the equations (2.11) together with the constraints of the form of Eq. (2.8). This one-to-one mapping ensures that the Principle of Determinacy also holds for generalized coordinates: The set of generalized coordinates and velocities $(q_1, \ldots, q_s; \dot{q}_1, \ldots, \dot{q}_s)$ at any point t completely determines the motion of a system.
☐ Solving a system with constraints with Newton's Laws. General scheme: Newton's Second Law for cartesian coordinates with all forces $\Longrightarrow$ Change to suitable generalized coordinates $\Longrightarrow$ Eliminate Forces of constraint $\Longrightarrow$ Effective equations of motion for generalized coordinates.
☐ Example: Frictionless pearl on rotating wire. A wire rotates with constant angular speed ω around the z-axis at an angle α with the axis. A pearl of mass m slides frictionlessly on the wire.

(i) Obviously the wire provides a geometric constraint for the motion of the pearl. Newton's Second Law including gravity and the force of constraint from the wire gives

$$m\ddot{\vec{r}} = -mg\hat{z} + \vec{Z}\,. \tag{2.12}$$

$\vec{Z}$ is not know at this stage.

(ii) We switch to better suited cylindrical coordinates r, ϕ and z. Note that we have two holonomic constraints. The wire forces $z - r\cot\alpha = 0$ and the motion of the wire in addition determines that $\phi - \omega t = 0$. Thus we have only one degree of freedom and we can choose the radial coordinate r as our one generalize coordinate. We use $x = r\cos\omega t$, $y = r\sin\omega t$ and $z = r\cot\alpha$. Plugging in to Newton's Law we obtain three equations of motion.

$$\ddot{r}\cos\omega t - 2\dot{r}\omega\sin\omega t - r\omega^2\cos\omega t = Z_x/m \qquad [1] \tag{2.13}$$
$$\ddot{r}\sin\omega t + 2\dot{r}\omega\cos\omega t - r\omega^2\cos\omega t = Z_y/m \qquad [2] \tag{2.14}$$
$$\ddot{r}\cot\alpha = Z_z/m - g \qquad [3] \tag{2.15}$$

(iii) Although we have only one degree of freedom we retain all three equations. In fact the total number of unknowns is four. We need one more equation to determine all unknowns. We need to find one more property of the force of constraint. We can get it be recalling that the force of constraint, like a normal force, has to be perpendicular to the wire, i.e.

$$Z_x\sin\alpha\cos\omega t + Z_y\sin\alpha\sin\omega t + Z_z\cos\alpha = 0 \quad [4]\,. \tag{2.16}$$

Taking $\sin\alpha\cos\omega t$ times [1] and adding $\sin\alpha\sin\omega t$ times [2], and using [4] we arrive at $\sin\alpha\ddot{r} - r\omega^2\sin\alpha = -\cos\alpha Z_z/m$. which we can use to eliminate Z_z in [3]. The final equation of motion for the generalized coordinate r is

$$\ddot{r}\,(\tan\alpha + \cot\alpha) - r\omega^2\tan\alpha + g = 0 \tag{2.17}$$

Of course we could go back and determine the force of constraint $\vec{Z}$ by the wire once we have a solution for $r(t)$.[3]

[3] We can easily solve the equation of motion since it is a linear ordinary differential equation with constant coefficients. We note that $r = g\cot\alpha/\omega^2$ is a particular solution of the inhomogeneous equation, while the solution

2.3 The Principle of Least Action

□ Newton's Second Law contains everything needed to solve any Mechanics problem. However, for more complicated problems switching to generalized coordinates and elimination of forces of constraint is tedious. We need something more effective to derive equations of motion.
□ Let a mechanical system be described by generalized coordinates $q = (q_1, \ldots, q_s)$ which take (initial and final) values $q^{(1)}$ and $q^{(2)}$ at particular times t_1 and t_2, respectively. The *Principle of Least Action* or *Hamilton's Principle* states: There exists a differentiable function

$$L(q, \dot{q}, t) = L(q_1, \ldots, q_s; \dot{q}_1, \ldots, \dot{q}_s; t) \tag{2.19}$$

for this system such that the motion between t_1 and t_2 leads to an extremum of the functional

$$S[q] = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \, . \tag{2.20}$$

L is called the Lagrangian or Lagrange function of the system and S is called the action of the system.
□ In the remainder of this section we will (i) reformulate the Priniciple of Least Action as a set of differential equations, (ii) set up a recipe to find the Lagrange function of a mechanical system, and (iii) show that the Principle of Least Action is equivalend to Newton's Second Law.

2.3.1 Lagrange Equations

□ For now suppose a Lagrange function L exists for a given system. Then $q(t)$ is the motion of the given system and satisfies the Principle of Least Action (with no constraints) exactly if

$$\sum_{i=1}^{s} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i = 0 \tag{2.21}$$

where the δq_i are variations of the coordinates around the extremum (and physical motion) q_i.
Proof: This follows immediately from Sec. (2.1) if the derivation of Eq. (2.4) is repeated for the functional $S[q_1, \ldots, q_s]$ depending on s real functions instead of just one. The expression on the left hand side of Eq. (2.21) is simply the integrand of the expression corresponding to Eq. (2.4) in this case, and Eq. (2.21) is a necessary and sufficient condition for $\delta S = 0$ at the extremum.
□ If a Lagrange function L exists and in addition all generalized coordinates are independent of each other (i.e. all constraints eliminated) then for above expression to vanish all terms in parentheses have to vanish independently. Thus we arrive at s independent equations of motion

$$\delta S = 0 \Longleftrightarrow \boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}} \quad \text{for } i = 1, \ldots, s \, . \tag{2.22}$$

This set of equations is known as the Lagrange equations of the second kind or simply the Lagrange equations.

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space of the homogeneous part is obvioulsy given by exponential functions. Hence the general solution is

$$r(t) = \frac{g}{\omega^2} \cot \alpha + A e^{\omega^2 \sin \alpha^2 t} + B e^{-\omega^2 \sin \alpha^2 t} \, . \tag{2.18}$$

A and B are determined by the initial conditions of the motion. An more intuitive way to think about this system is via an effective potential energy for the generalized coordinate r (see Sec. ...) which can be obtained from integrating the "generalized force" $m\ddot{r}$: $U(r) = -\frac{m}{2}\omega^2 r^2 \sin^2 \alpha + mgr \sin \alpha \cos \alpha$. We see that the stationary particular solution corresponds to orbits with the unstable equilibrium point in the effective potential energy as the radius.

2.3.2 The Lagrange Function

Before we can start solving mechanical problems with Lagrange equations we still need to make sure we can find a Lagrange function $L(q, \dot{q}, t)$. First we establish that Lagrange functions are not unique.

☐ Gauge transformations: The Lagrange functions

$$L(q, \dot{q}, t) \quad \text{and} \quad L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \tfrac{d}{dt} f(q, t) \tag{2.23}$$

are equivalent for any differentiable function $f(q, t)$. With equivalent we mean that they give the same solution for the Principle of Least Action, i.e. $\delta S = 0 \Longleftrightarrow \delta S' = 0$ where

$$S = \int_{t_1}^{t_2} L dt \quad , \quad S' = \int_{t_1}^{t_2} L' dt \tag{2.24}$$

for arbitrary t_1, t_2. Such transformations with total time derivatives are called (mechanical) gauge transformations.

Proof:

$$S'[q] = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt + \int_{t_1}^{t_2} \frac{d}{dt} f(q, t) dt = S[q] + f(q_2, t_2) - f(q_1, t_1) \tag{2.25}$$

The two last terms vanish under variations since they are evaluated at the boundaries. One can also work out the proof using the Lagrange equations.

☐ With one Lagrange function known we can construct infinitely many through gauge transformations. We also note that the Lagrange function itself can not be a physical observable since it is ambiguous.

☐ We can use Galilei invariance to construct the Lagrange function of a free particle. Note that L does not have to be strictly invariant under Galilei transformations. Rather, a Galilei transformation is permitted to change L into another Lagrange function that is related to the original via a gauge tranformation. (The motion determined by those two Lagrangians would be the same!)

Using first translational invariance in time and space and then rotational invariance we note that for a single free mass

$$L(\vec{r}, \vec{v}, t) \equiv L(\vec{v}) \equiv L(v^2) \, . \tag{2.26}$$

Now let us investigate how $L(v^2)$ changes from a frame of reference K to another frame K' which moves with infinitesimally small velocity $d\vec{w}$ with respect to K. The Lagrange function in K' is

$$L' = L\left((\vec{v} + d\vec{w})^2\right) = L(v^2) + 2\frac{dL}{dv^2} \vec{v} \cdot d\vec{w} + \mathcal{O}\left((dw)^2\right) \tag{2.27}$$

Dropping terms beyond the first order in dw we require L' to be equal to L up to a gauge transformation. For that we need the second term to be the total time derivative of a function. Now $\vec{v}$ already is a time derivative so dL/dv^2 should be a constant. Then

$$L' = L + \frac{d}{dt}\left(2\frac{dL}{dv^2} \vec{r} \cdot d\vec{w}\right) \, . \tag{2.28}$$

Hence we conclude that $L = \text{const.} \times v^2$. We are free to choose the constant and define "the"[4] Lagrange function of a free point mass m to be equal to its kinetic energy,

$$L = \frac{1}{2} m v^2 = T \, .^5 \tag{2.29}$$

[4] Of course we can find infinitely more through gauge tranformations.

[5] One can show that L' and L are also equivalent if related through a finite velocity $\vec{w}$, see LL, §4.

☐ Next consider a point particle subject to a conservative force $\vec{F}(\vec{r}) = -\nabla U(\vec{r})$. By comparing the Lagrange Equations with Newton's Second Law we find that

$$L = \frac{1}{2}mv^2 - U(\vec{r}) \tag{2.30}$$

is an acceptable Lagrange function in that case. Proof:

$$\frac{\partial L}{\partial r_i} - \frac{d}{dt}\frac{\partial L}{\partial v_i} = 0 \Longleftrightarrow F_i - m\dot{v}_i = 0 \tag{2.31}$$

☐ The general case: Let a system be described by generalized coordinates $q_1, \ldots, q_s$ without further holonomic constraints, with kinetic energy $T(q_1, \ldots, q_s; \dot{q}_1, \ldots, q_s)$ and kinetic energy $U(q_1, \ldots, q_s)$. Then

$$\boxed{L = T - U} \tag{2.32}$$

is a Lagrange function of the system. Proof: : Will follow from considerations in the next chapter.
☐ Let us discuss Lagrange functions in generalized coordinates further, starting from their cartesian counterpart $L(x, \dot{x}, t)$ for the same system. Since $x_i = x_i(q_1, \ldots, q_s)$ (note the missing explicit time dependence) we have

$$\dot{x}_i = \sum_{j=1}^{s} \frac{\partial x_i}{\partial q_j}\dot{q}_j \,. \tag{2.33}$$

The Lagrange function $T - U$ can be expressed as

$$L(q, \dot{q}, t) = \frac{1}{2}\sum_{k,l=1}^{s} M_{kl}(q)\dot{q}_k\dot{q}_l - U(q) \quad \text{with} \quad M_{kl} = \sum_{i=1}^{N} \frac{\partial x_i}{\partial q_k}\frac{\partial x_i}{\partial q_l} \,. \tag{2.34}$$

In particular, the kinetic energy is still a quadratic form in generalized velocities (but in general no longer diagonal and it can also explicitly depend on coordinates).[6]
☐ Some important special cases:

- Cylindrical coordinates r, ϕ, z. The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) \,. \tag{2.35}$$

- Spherical coordinates r, θ, ϕ. The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \,. \tag{2.36}$$

☐ We revist the example of the frictionless pearl on a rotating wire from the end of Sec. 2.2. Instead of using Newton's Law we use generalized coordinates and Lagrange equations and we will enjoy to see how much simplified the calculation becomes.

(i) We set up a cylindrical coordinate system and with only one degree of freedom we opt for r as our one generalized coordinate.

[6]You can check that if we allowed explicit time dependences in the transformation to generalized coordinates the kinetic energy would no longer be purely quadratic in velocities.

(ii) The Lagrange function,using the constraint $z = r\cot\alpha$ is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\omega}^2 + \dot{r}^2\cot\alpha\right) - mgr\cot\alpha\,. \tag{2.37}$$

(iii) The Lagrange equation is

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r}\left(1 + \cot^2\alpha\right) - mr\omega^2 + mg\cot\alpha \tag{2.38}$$

which is the same as Eq. (2.17) obtained with much less work.

2.3.3 The d'Alembert Principle

☐ The d'Alembert Principle is a "differential" version of the Principle of Least Action. Consider a system of particles at positions $\vec{r}_i$, $i = 1,\ldots,N$, subject to forces of constraint $\vec{Z}_i$ and driving forces $\vec{F}_i$. Then $\dot{\vec{p}} = \vec{F}_i + \vec{Z}_i$. Further consider an arbitrary small variation $\delta\vec{r}_i$ of each particle *at a fixed time* which is *consistent with the constraints*. We postulate

$$\sum_{i=1}^{N} \vec{Z}_i \cdot \delta\vec{r}_i = 0\,. \tag{2.39}$$

In words, forces of constraint do not do work for variations of positions that are consistent with the constraints. This is clear from our definiton of allowed variations.
☐ Using Newton's Second Law we can rewrite the condition above as

$$\boxed{\sum_{i=1}^{N}\left(\vec{F}_i - \dot{\vec{p}}_i\right)\cdot\delta\vec{r}_i = 0}\,. \tag{2.40}$$

This is known as *d'Alembert's Principle*. It is clear that the d'Alembert Principle is equivalent to Newton's Second Law. In the following we will show that d'Alembert's Principle is equivalent to the Principle of Least Action for problems for which a Lagrange function exists, and hence Newton's Second Law is equivalent to the Principle of Least Action in that case as well:

$$\boxed{\text{Newton}} \Longleftrightarrow \boxed{\text{d'Alembert}} \underset{L\text{ exists}}{\Longleftrightarrow} \boxed{\text{Hamilton}} \tag{2.41}$$

☐ Generalized Forces: Let $q = (q_1,\ldots,q_s)$ be generalized coordinates for the system. We define the *generalized forces* Q_j, $j = 1,\ldots,s$ (from the driving forces) as

$$Q_j = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial\vec{r}_i}{\partial q_i} \tag{2.42}$$

Depending on the dimension of q_j the dimension of Q_j is not necessarily that of a "cartesian" force. E.g. if q_j is an angle Q_j will be a torque.
☐ For conservative systems $\vec{F}_i = -\nabla_i U$. In that case

$$\boxed{Q_j = -\frac{\partial U}{\partial q_j}} \tag{2.43}$$

for $j = 1, \ldots, s$. Proof:

$$Q_j = -\sum_{i=1}^{N} (\nabla_i U) \cdot \frac{\partial \vec{r}_i}{\partial q_i} = -\frac{\partial U}{\partial q_j} \,. \tag{2.44}$$

□ Next we rewrite d'Alembert's Principle in generalized coordinates. The term involving the forces is straight forward to express in generalized forces

$$\sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i = \sum_{i=1}^{N}\sum_{j=1}^{s} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{s} Q_j \delta q_j \tag{2.45}$$

where the δq_j are equal time variations of the generalized coordinates which are consistent with the constraints. On the other hand we have

$$\sum_{i=1}^{N} \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_{j=1}^{s} \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right) \delta q_j \tag{2.46}$$

where T is the kinetic energy. Proof: See Homework.

We conclude that d'Alembert's Principle is equivalent to the statement

$$\sum_{j=1}^{s} \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0 \,. \tag{2.47}$$

□ For the last step in our proof to show the equivalency of d'Alembert and the Principle of Least Action we restrict ourselves to conservative systems, i.e. we have $Q_j = -\partial U/\partial q_j$ and a Lagrange function $L = T - U$ exists. Then d'Alembert is equivalent to

$$\sum_{j=1}^{s} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} \right) \delta q_j = 0 \tag{2.48}$$

which is the same as Eq. (2.21). We note that the same arguments go through for those $non-conservative$ systems that do possess a Lagrange function. We will discuss those cases in the next subsection. Q.E.D.

Of course we reclaim the set of s Lagrange equations from the last equation if all variations δq_j are independent of each other, i.e. all constraints are eliminated in the generalized coordinates.

2.3.4 Non-Conservative Systems

□ The d'Alembert Principle provides a method that allows us to write down equations of motion for systems with generalized forces Q_i independent of whether or not these forces are conservative. For holonomic systems with s independent coordinates we have the set of equations

$$\boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i} \tag{2.49}$$

for $i = 1, \ldots, s$.

□ We can allow the following generalization of a potential energy function which allows us to keep the same Lagrange equations we have for conservative systems. We call $U(q, \dot{q})$ a *generalized potential energy* for the system if

$$\boxed{Q_j = \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} - \frac{\partial U}{\partial q_j}} \tag{2.50}$$

for all $j = 1, \ldots, s$. Note that conservative forces are included in this definition as a special case. Then we define $L = T - U$ and obviously the d'Alembert Princple is equivalent to the set of Lagrange equations using this Lagrange function L.

☐ An important example: the electromagnetic force. Recall that for external electric fields $\vec{E}$ and $\vec{B}$ the force on a charge q moving with velocity $\vec{v}$ is $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. Because of the velocity dependence the electromagnetic force in general is non-conservative. Let us introduce the scalar potential ϕ and vector potential $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla\phi - \frac{\partial}{\partial t}\vec{A}$.[7] The generalized potential for the electromagnetic force is

$$U = q\left(\phi - \vec{v} \times \vec{A}\right) \tag{2.51}$$

Proof: See HW.

☐ A special case: the friction force. Friction is an important non-conservative force for which no generalized potential exists. Friction forces linear in velocity can be implemented into the Lagrange formalism using a special construction, the so called dissipative function. Let $Q_j = Q_j^{(U)} + Q_j^{(R)}$, $j = 1, \ldots, s$ be a decomposition of the generalized forces into a part for which a generalized potential U exists and friction forces $Q_j^{(R)}$. d'Alembert's Principle tells us that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^{(R)} \tag{2.52}$$

for $j = 1, \ldots, s$. No assume that the friction foces are linear in the velocities, $Q_j^{(R)} = -\sum i = 1^s \beta_{ji}\dot{q}_i$ with coefficients $\beta_{ji} = \beta_{ij}$ given by phenomenology.[8] We define *Rayleigh's dissipative function* as

$$\boxed{D = \frac{1}{2}\sum_{i,j=1}^{s} \beta_{ij}\dot{q}_i\dot{q}_j}\,. \tag{2.53}$$

Obviously we have

$$Q_j = -\frac{\partial D}{\partial \dot{q}_j} \quad \text{and} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = 0\,. \tag{2.54}$$

The latter equation is called a modified Lagrange equation (of the second kind). D is another scalar function describing the system besides L.

☐ For systems with Lagrange function $\partial L/\partial t = 0$ and a dissipative function D for which only contains conservative forces contribute to L we have the following equation for the rate of energy loss:

$$\frac{dE}{dt} = -2D\,. \tag{2.55}$$

Proof: Expanding the time derivative on $T + U$ using the chain rule and using partial integration on the velocity derivatives of T we obtain

$$\begin{aligned} \frac{dE}{dt} &= \sum_{j=1}^{s}\left(\frac{\partial T}{\partial q_j}\dot{q}_j + \frac{\partial T}{\partial \dot{q}_j}\ddot{q}_j\right) + \frac{dU}{dt} = \sum_{j=1}^{s}\left(\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial U}{\partial q_j}\dot{q}_j\right) + \sum_{j=1}^{s}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j\right) - \dot{q}_j\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}\right] \\ &= \sum_{j=1}^{s}\dot{q}_j\left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right) + 2\frac{dU}{dt} + 2\frac{dT}{dt} = \sum_{j=1}^{s}\dot{q}_j\frac{\partial D}{\partial \dot{q}_j} + 2\frac{dE}{dt} = 2D + 2\frac{dE}{dt} \quad \text{Q.E.D.} \end{aligned} \tag{2.56}$$

[7] Consult your favourite book on Electromagnetism if you're not sure about this.

[8] Friction forces are not fundamental in nature.

2.3.5 Non-Holonomic Systems

□ Consider systems with coordinates $q_1, \dots, q_s$ and p conditions of the type

$$\sum_{i=1}^{N} \omega_i^j dq_i + \omega_0^j dt = 0\,, \tag{2.57}$$

$j = 1, \dots, p$. Hence for variations δq_i at equal time

$$\sum_{i=1}^{s} \omega_i^j \delta q_i = 0 \tag{2.58}$$

for $j = 1, \dots, p$ and trivially the linear combination of those equations

$$\sum_{j=1}^{p} \lambda_j \sum_{i=1}^{s} \omega_i^j \delta q_i = 0 \tag{2.59}$$

is true for any parameters λ_j, $j = 1, \dots, p$. The λ_j are called Lagrange multipliers.
□ The d'Alembert Principle (2.48) and the previous equation taken together tell us that

$$\sum_{i=1}^{s} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{p} \lambda_j \omega_i^j \right) \delta q_i = 0 \tag{2.60}$$

where the δq_i are not independent because of the differential constraints. However we are free to choose a subset of them, say $\delta q_1, \dots, \delta q_{s-p}$ independently and then have $q_{s-p+1}, \dots, \delta q_s$ determined by the constraints.

In order to break up the previous long equation into s independent equations we use the fact that we can choose the Lagrange parameters any way we want. We opt to have the p parameters λ_i satisfy the p equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{p} \lambda_j \omega_i^j = 0 \tag{2.61}$$

for $i = s-p+1, \dots, s$. Then Eq. (2.60) reduces to the smaller equation

$$\sum_{i=1}^{s-p} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{p} \lambda_j \omega_i^j \right) \delta q_i = 0 \tag{2.62}$$

where are the remaining δq_i are independent. Thus

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{p} \lambda_j \omega_i^j = 0 \tag{2.63}$$

for $i = 1, \dots, s-p$.

In summary this means that we have $s+p$ equations to solve for the mechanical problem under discussion. We have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{p} \lambda_j \omega_i^j \tag{2.64}$$

for $i = 1, \ldots, s$ which are called the *Lagrange equations of the first kind*, and p equations

$$\sum_{i=1}^{s} \omega_i^j \dot{q}_i + \omega_0^j = 0 \tag{2.65}$$

for $j = 1, \ldots, p$ from the differential constraints.

☐ Let us compare Eqs. (2.64) and (2.47). Clearly the

$$\bar{Q}_i = \sum_{j=1}^{p} \lambda_j \omega_i^j \tag{2.66}$$

$i = 1, \ldots, s$ are generalized forces related to the constraints.

☐ Note that we can use Lagrange equations of the first kind for holonomic constraints as well if we are interested in calculating the forces of constraint.

☐ Example: The frictionless Atwood Machine. Consider a (massless,unstretchable) rope of lengh l running over a massless pulley (radius R) with two masses m_1 and m_2 attached to the two ends of the rope. What is the acceleration of the system and what is the force in the rope.

Let z_1 and z_2 be the length of either side of the rope as measured from the height of the center of the pulley straight down towards masses m_1 and m_2 respectively. The constraint from the rope is

$$z_1 + z_2 + \pi R - l = 0 \tag{2.67}$$

Obviously we could eliminate this holonomic constraint but since we are interested in calculating the force of constraint we keep both z_1 and z_2 as coordinates.

By differentiating the constraint we obtain

$$\dot{z}_1 + \dot{z}_2 = 0 \quad \text{or} \quad dz_1 + dz_2 = 0\,. \tag{2.68}$$

Thus $\omega_1 = 1 = \omega_2$.[9] The Lagrange function is

$$L = \frac{m_1}{2}\dot{z}_1^2 + \frac{m_2}{2}\dot{z}_2^2 + m_1 g z_1 + m_2 g z_2 \tag{2.69}$$

The Lagrange equations of the first kind read

$$m_1 \ddot{z}_1 - m_1 g = \lambda \qquad , \qquad m_2 \ddot{z}_2 - m_2 g = \lambda \tag{2.70}$$

Obviously $\bar{Q}_1 = \lambda = \bar{Q}_2$ is the force of tension acting from the rope onto both masses.

From the constraint we know $\ddot{z}_1 = -\ddot{z}_2$ and after eliminating λ we obtain

$$\ddot{z}_1 = -\ddot{z}_2 = g\frac{m_1 - m_2}{m_1 + m_2}\,. \tag{2.71}$$

Hence the tension is

$$\lambda = -2g\frac{m_1 m_2}{m_1 + m_2} \tag{2.72}$$

which is negative (i.e. upward) with our choice of coordinates.

2.4 Noether's Theorem and Integrals of Motion

In this section all systems shall be holonomic and conservative unless stated otherwise.

[9] We omit the upper index since there is only one constraint.

2.4.1 Integrals of Motion

☐ For a system with generalized coordinates $q_1, \ldots, q_s$ and Lagrange function L we define the generalized momenta

$$\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j}} \tag{2.73}$$

)$j = 1, \ldots, s$).

Examples:

- For a free particle $L = \frac{m}{2}\dot{r}^2$ we have $p_j = \partial L/\partial \dot{r}_j = m\dot{r}_j$ as expected.
- For a particle in and electromagnetic field $L = \frac{m}{2}\dot{r}^2 - q\phi + q\dot{\vec{r}} \cdot \vec{A}$ we have $p_j = m\dot{r}_j + qA_j$ which is not identical to the mechanical momentum $m\dot{r}_j$.

☐ The coordinate q_j is called cyclical if and only if $\frac{\partial L}{\partial q_j} = 0$, i.e. the system does not depend on q_j. Then

$$\boxed{q_j \text{ cyclical } \Leftrightarrow p_j = \text{const.}} \tag{2.74}$$

Proof: $\frac{\partial L}{\partial q_j} = 0$ in the Lagrange equation implies $\frac{d}{dt}(\partial L/\partial \dot{q}_j) = 0$. Conservation laws like this are often useful for solving equations of motion.

☐ Suppose for the system under consideration there exist certain functions $F_k(q_1, \ldots, q_s; \dot{q}_1, \ldots, \dot{q}_s; t)$, $k = 1, \ldots, r$ with $r \leq 2s$ such that all $F_k = \text{const.}$ for the motion. Such functions F_k (and their constant values) are called integrals of motion.

Example: Generalized momenta p_k of cyclical coordinates q_k are integrals of motion.

☐ General strategy: try to find a set of generalized coordinates with as many cyclical coordinates as possible to make a problem easier.

Example: Two masses m_1, m_2 with interaction given by potential $U(|\vec{r}_2 - \vec{r}_1|)$. Choose center of mass coordinates $q_1 = X$, $q_2 = Y$, $q_3 = Z$ and a spherical representation of the relative coordinate $\vec{r}$: $q_4 = r$, $q_5 = \theta$, $q_6 = \phi$. Then

$$L = \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - U(r) \tag{2.75}$$

and

$$X \text{ cycl.} \Longrightarrow p_1 = M\dot{X} = P_x = \text{const.} \tag{2.76}$$

$$Y \text{ cycl.} \Longrightarrow p_2 = M\dot{Y} = P_y = \text{const.} \tag{2.77}$$

$$Z \text{ cycl.} \Longrightarrow p_3 = M\dot{X} = P_z = \text{const.} \tag{2.78}$$

$$\phi \text{ cycl.} \Longrightarrow p_6 = \mu r^2\sin^2\theta\dot{\phi} = (L_r)_z = \text{const.}. \tag{2.79}$$

Here $\vec{L}_r = \mu\vec{r} \times \dot{\vec{r}}$ is the angular momentum vector associated with the relative coordinate, cf. Eq. (1.66). From the absence of a torque we can furthermore conclude that the entire vector $\vec{L}_r$ is conserved.

2.4.2 Noether's Theorem

☐ Let $L(q, \dot{q})$ be the Lagrange function of a system with motion $q(t) = (q_1(t), \ldots, q_s(t))$ which is a map $q: I \longrightarrow K$ from a time interval $I \subset \mathbb{R}$ into the s-dimensional configuration space $K \subset \mathbb{R}^s$

spanned by the generalized coordinates q_i[10]. Next we define continuous transformations of the motion (think of coordinate transformations as examples). In order to be general we do this by introducing a map $h : J \times I \longrightarrow K$, $(u,t) \mapsto h(u,t)$ where the parameter u is from an open interval $J \subset \mathbb{R}$ that contains zero. We say that h is a transformation of the motion q if it satisfies the following properties.

1. $h(0,t) = q(t)$
2. $h(u,t)$ is differentiable in u and twice differentiable in t.
3. $L\left(h(u,t), \dot{h}(u,t)\right) = L(q(t), \dot{q}(t))$ for all values of u in J.

In other words, we have a continuum of possible motions, parameterized by u that lead to the same Lagrange function as a function of time t.[11]

□ Noether's Theorem. For a system with a transformation h as defined above

$$\boxed{F_h(q,\dot{q}) = \sum_{j=1}^{s} \frac{\partial L}{\partial \dot{q}_j} \frac{d}{du} h_j(u,t)\Big|_{u=0}} \tag{2.80}$$

is an integral of motion.

Proof: Because of the invariance of the Lagrange function under transformations we have

$$0 = \frac{d}{du} L\left[h(u,t), \dot{h}(u,t)\right]\Big|_{u=0} = \sum_{j=1}^{s} \left[\frac{\partial L}{\partial q_j} \frac{dh_j}{du}\Big|_{u=0} + \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{h}_j}{du}\Big|_{u=0} \right] \tag{2.81}$$

Applying the Lagrange equations on the first term and exchanging derivative in the second term we arrive at

$$0 = \sum_{j=1}^{s} \left[\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \frac{dh_j}{du}\Big|_{u=0} + \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \frac{dh_j}{du}\Big|_{u=0} \right] = \frac{d}{dt} \sum_{j=1}^{s} \left[\frac{\partial L}{\partial \dot{q}_j} \frac{dh_j}{du}\Big|_{u=0} \right]. \tag{2.82}$$

2.4.3 Integrals of Motion Related to Symmetries of Space and Time

In this section we connect the already known fundamental integrals of motion, energy, momentum and angular momentum, to space-time symmetries. Along the way we learn how to apply the Noether Theorem in practice.

Energy

□ We start with energy which is actually the one which is outside of Noether's Theorem as we have just proven it. Let us look at a holonomic system with generalized coordinates $q_1, \ldots, q_s$ for which a Lagrange function L exists (we do not demand the system to be conservative for this discussion). We define the *Hamilton function* H for a system as

$$\boxed{H = \sum_{j=1}^{s} p_j \dot{q}_j - L} \tag{2.83}$$

[10]If the coordinates are all cartesion $K = \mathbb{R}^s$ but we leave the possibility that some generalized coordinates have naturally "smaller" allowed range.

[11]Don't be confused here, the motion of a system is still uniquely determined once initial conditions are fixed which is not required here!

where the p_j are the generalized momenta.

☐ A system is homogeneous in time, if and only if the Hamilton function is an integral of motion, i.e.

$$\boxed{\frac{\partial L}{\partial t} \Longleftrightarrow H = \text{const.}} . \tag{2.84}$$

Proof: This follows from Beltrami's Identity discussed in the homework, but we give an explicit proof here nevertheless. We have

$$\begin{aligned} \frac{d}{dt}H &= \sum_{j=1}^{s}\left[\frac{d}{dt}(p_j\dot{q}_j) - \frac{\partial L}{\partial q_j}\dot{q}_j - \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right] - \frac{\partial L}{\partial t} \\ &= \sum_{j=1}^{s}\left[\frac{d}{dt}(p_j\dot{q}_j) - \frac{\partial L}{\partial q_j}\dot{q}_j + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j - \frac{d}{dt}(p_j\dot{q}_j)\right] - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t} . \end{aligned} \tag{2.85}$$

☐ Now consider a system where in addition all constraints are independent of time (i.e. *scleronomic*). In that case the Hamilton function measures the total energy of the system

$$\boxed{H = T + U = E} \tag{2.86}$$

Proof: Scleronomic conditions mean that the generalized coordinates do not depend explicitly on time. Hence the kinetic energy can be written as discussed in Eq. (2.34),

$$T = \frac{1}{2}\sum_{i,j=1}^{s} M_{ij}(q)\dot{q}_i\dot{q}_j \tag{2.87}$$

and

$$\sum_{j=1}^{s} p_j\dot{q}_j = 2T \quad \Longrightarrow \quad H = T + U . \tag{2.88}$$

☐ In particular we can deduce the following statement: in a holonomic-scleronomic system with $\partial L/\partial t = 0$ the total energy $E = H$ is conserved.

☐ Example for a system with Hamilton function conserved but energy not conserved: pearl on a rotating wire.

Momentum

☐ Consider a system of n particles with $N = 3n$ degrees of freedom at coordinates $x = (\vec{x}_1, \ldots, \vec{x}_n)$. We return to the case of holonomic and conservative systems for the rest of this chapter. Let $\hat{e}$ be a unit vector in $\mathbb{R}^3$ and $h(u,t)$ be a transformation on the trajectory $x(t)$ that realizes translations in space along the direction $\hat{e}$, i.e.

$$h(u,t) = x(t) + u\hat{e}^n \tag{2.89}$$

where $\hat{e}^n$ is the vector $(\hat{e}, \ldots, \hat{e})$ with n entries. Obviously h satisfies the first two conditions for a transformation from Noether's Theorem.

☐ Let us now consider a system which is translationally invariant in space, i.e. a system for which $x(t)$ is a solution (without imposing boundary conditions) if and only if $x(t) + a$ is a solution for any N-dimensional vector a. In that case condition (3) for a transformation eligible for Noether's theorem is satisfied as well and we know that

$$\sum_{j=1}^{N} \frac{\partial L}{\partial \dot{x}_j} \frac{d}{du} h_j(u,t)\Big|_{u=0} = \sum_{i=1}^{n} \vec{p}_i \cdot \hat{e} = \vec{P} \cdot \hat{e} \tag{2.90}$$

is an integral of motion. Here the $\vec{p}_i$ are the momenta of the individial particles and $\vec{P}$ is the total momentum of the system. We note that the derivative in the derivation is

$$\frac{d}{du} h(u,t)\Big|_{u=0} = \hat{e}^n \,. \tag{2.91}$$

Thus we obtain that the total momentum of the system projected onto the direction $\hat{e}$ is conserved. But of course $\hat{e}$ was arbitrality chose, so we conclude that for a translationally invariant system the total momentum $\vec{P}$ is conserved, as expected. Comparing with the old statement about momentum conservation we note that the translational invariance requires external forces to vanish.

□ In particular for a single particle we reconfirm the previous statement that the momentum component p_j is conserved if and only if the Lagrange function does not depend explicitly on the coordinate x_j, i.e. x_j is cyclical.

Angular Momentum

□ Consider a system of n particles as above. Let $\hat{e}$ be an axis and ϕ a rotation angle around $\hat{e}$ (counted according to the right hand rule). We define the vector $\vec{\phi} = \phi\hat{e}$ to describe this rotation. For a point $\vec{r}$ the image under a small rotation $\Delta\vec{\phi}$ is $\vec{r} + \Delta\vec{r}$ where $\Delta\vec{r} = \Delta\vec{\phi} \times \vec{r} = \Delta\phi\hat{e} \times \vec{r}$. We define a transformation $h(\Delta\phi, t)$ as a rotation that maps $\vec{x}_i(t)$ into $\vec{x}_i(t) + \Delta\phi\hat{e} \times \vec{x}_i(t)$ for all $i = 1, \ldots, n$. Again we observe that conditions (1), (2) for the transformation being eligible for Noether's Theorem are fulfilled.

□ Now we consider only those systems that are invariant under such rotations for any $\hat{e}$, i.e. the system is isostropic. Noether's Theorem applies and we know that

$$\sum_{j=1}^{N} \frac{\partial L}{\partial \dot{x}_j} \frac{d}{d\Delta\phi} h_j(\Delta\phi,t)\Big|_{\Delta\phi=0} = \sum_{i=1}^{n} \vec{p}_i \cdot (\hat{e} \times \vec{x}_i) = \vec{L} \cdot \hat{e} \tag{2.92}$$

is an integral of motion, where $\vec{L}$ is the total angular momentum of the sustem with respect to the origin. Hence the component of the angular momentum along $\hat{e}$ is conserved but if rotational invariance is true for any orientation of $\hat{e}$ the total angular momentum vector $\vec{L}$ is conserved.

Examples

□ Example 1: A particle moves in the field of an infinite, homogeneous plane. What components of $\vec{p}$ and $\vec{L}$ are conserved?

Let $U(\vec{r})$ be the potential energy of the particle, choose coordinates such that the plane coincides with the $x-y$-coordinate plane. We have translational invariance in x- and y-direction, $\partial U/\partial x = 0 = \partial U/\partial y$ and therefore p_x and p_y are conserved. $U(\vec{r})$ is also invariant under rotations around the z-axis, $\partial U/\partial\phi = 0$ with $\vec{\phi} = \phi\hat{z}$, thus $L_z = \hat{z} \cdot \vec{L}$ is conserved.

□ Example 2: A particle moves in the field of two point charges. What components of $\vec{p}$ and $\vec{L}$ are conserved?

Let $U(\vec{r})$ be the potential energy, choose the z-axis through both point charges. $U(\vec{r})$ is also invariant under rotations around the z-axis, $\partial U/\partial\phi = 0$ with $\vec{\phi} = \phi\hat{z}$, thus $L_z = \hat{z} \cdot \vec{L}$ is conserved.