Asymptotic normalization coefficients from $^3He + ^4He$ elastic scattering and astrophysical factor for $^3He + ^4He \rightarrow ^7Be + \gamma$

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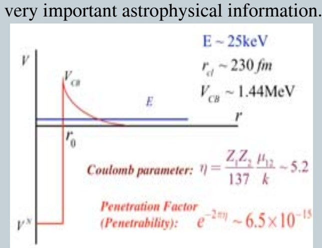
Problem:

- Nuclear astrophysical reactions occur at very low energies.
- Due to the presence of the Coulomb barrier, the astrophysical cross sections are extremely small. Therefore, they are very difficult, or even impossible, to measure.
- We need different indirect methods of obtaining the astrophysical S factor.

Goal:

- ***The goal of my project is to present an indirect method of calculating the astrophysical S factor based on the information obtained from the elastic scattering.***
- Specifically, I analyze the elastic scattering of $^3He + ^4He$ to obtain the astrophysical S factor for $^3He + ^4He \rightarrow ^7Be + \gamma$.
- By addressing this elastic scattering indirect technique, we hope that more accurate measurements of elastic scattering data will provide very important astrophysical information.



Previous Attempts:

- The elastic scattering phase shifts have been used before to get astrophysical information: P. Mohr arXiv:0906.3000(2009), P. Mohr et. el. Phys. Ref. C 48 1420 (1993).
- They searched for the two-body $^3He + ^4He$ potential that could fit the phase shifts and then calculated the S factor (based on the "fitted" potential).
- From there, the "fitted" potential was used to calculate the bound state wavefunction of $^3He + ^4He$.
- However, since the reaction of $^3He + ^4He \rightarrow ^7Be + \gamma$ is **peripheral**, all that is needed to find the S factor is the ANC.
- But, according to the inverse scattering theorem (Gel'fand, et al.), there is an infinite number of phase equivalent potentials which yield different ANCs.

Our Approach:

- Instead of using the potential model, we use the analyticity of the elastic scattering S matrix to interpolate the available experimental phase shifts in the physical region and extrapolate to the pole of the S matrix corresponding to the bound states.
- Since the residue in the pole of the S matrix is proportional to the square of the ANC, we are then able to determine the value of the ANC for $^3He + ^4He \rightarrow ^7Be$ from extrapolation of the S matrix to the bound state pole.
- Taking into account that the $^3He + ^4He \rightarrow ^7Be + \gamma$ reaction is peripheral, we can calculate the astrophysical factor for this reaction down to zero energy.

Method:

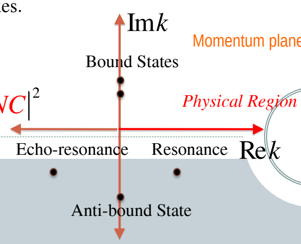
- We know that the elastic scattering S matrix is an analytical function in the momentum plane:

$$S_l(k) = e^{2i\delta_l(k)}$$

- Thus, the S matrix has derivatives of all orders everywhere except for the poles corresponding to the bound, anti-bound (virtual) states, resonances, and the cuts along positive and negative imaginary axes.

$$S_l(k) = \frac{A_l}{k - i\kappa}$$

$$A_l = (-1)^{l+1} i |ANC|^2$$



- Bound state wavefunction at $r > r_0$ is given by:

$$\varphi_l(r) \sim ANC \cdot W_{\eta_\kappa,\, l+\frac{1}{2}}(\kappa r) \sim ANC\,\frac{e^{-\kappa r}}{r^{\eta_\kappa}}$$

$W(\kappa r)$ is the Whittaker hyper-geometric function, $\eta_\kappa = \frac{Z_1 Z_2 e^2 \mu_{12}}{\kappa}$, is the Coulomb parameter for the bound state and $\kappa = \sqrt{2\mu_{12}\varepsilon}$ is the bound state wave number.

- The scattering wavefunction at $r > r_0$ is given by:

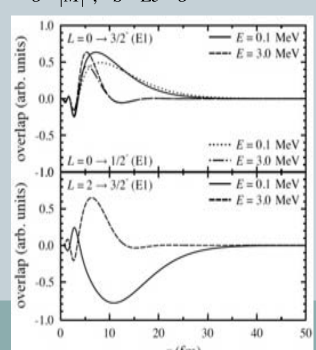
$$\chi_l(r) \sim C_0[F(r)G(r_0) - G(r)F(r_0)]$$

F(r) is the regular solution of the radial Schrodinger equation with Coulomb interaction and G(r) is the singular solution.

- The matrix element, which determines the amplitude for the E1 (electric dipole) radiative capture $l = 0 \rightarrow l = 1$, is proportional to the integral:

$$M \sim \int_{r_0}^{\infty} dr\, r\varphi_1(r)\chi_0(r) \sim ANC \int_{r_0}^{\infty} dr\, r W_{\eta_\kappa,\frac{3}{2}}(\kappa r)\chi_0(r)$$

$$\sigma \sim |M|^2, \quad S \sim E e^{2\pi\eta}\sigma$$



Method (cont'd):

- To calculate this matrix element, we have all of the components except for the value of the ANC.
- We want to obtain this value by analyzing the experimental phase shifts for $l = 1$.
- We know that the renormalized scattering amplitude for charged particles $F_l(k)$ can be written as:

$$F_l(k) = \frac{S_l(k) - 1}{2ik\rho_l(k^2)} = \frac{1}{k\cot(\delta_l(k)) - ik}\frac{1}{\rho_l(k^2)}$$

(Mukhamedzhanov et al. 1984)

*where $\delta_l(k) = \delta_{tot}(k) - \sigma_l(k)$ is the **Coulomb-modified** nuclear phase shift (subtracting the known pure Coulomb scattering phase shift from the total phase shift), and $\rho_l(k) = kC_l^2$ is the renormalization factor for the scattering amplitude.*

- Since we consider the low-energy region, we can use the effective range theory so that $k\cot(\delta_l(k)) - ik$ can be replaced by:

$$\left[\frac{K_l(k^2)}{k^{2l}\prod_{n=1}^{l}\left(1 + \frac{\eta^2}{n^2}\right)} - 2\eta k \cdot h(\eta)\right]\frac{1}{C_0^2(\eta)}$$

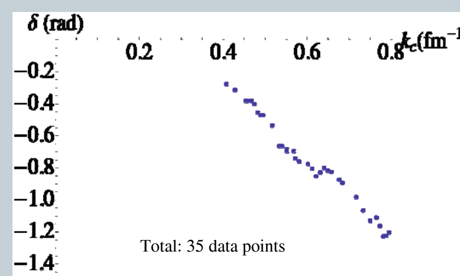
where $C_0^2(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1}$ is the penetration factor for l = 0, and $\eta = \frac{Z_1 Z_2}{137}\frac{\mu_{12}}{k}$.

- $K_l(k^2)$ is the effective range function, which is an analytical function of k^2. We approximate it by the Padé form:

$$K_l(k^2) = \frac{P_n(k^2)}{Q_m(k^2)} = \frac{a_1 + a_2 k^2 + a_3 k^4 + \ldots}{1 + b_1 k^2 + b_2 k^4 + \ldots}$$

Data Analysis:

We combined two sets of data: Hardy (NP A 1972) P (3/2 -) and Boykin (NP A 1972) P(3/2-).



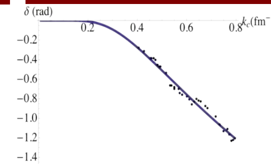
First Interpolation:

First, we fit the elastic scattering phase shift using the Padé approximation:

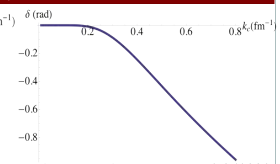
$$\delta_l(k) = -\frac{2\pi}{(l!)^2}\xi^{2l+1}\frac{1}{e^{\frac{2\pi\xi}{kx}} - 1}\frac{a_1 + a_2 k^2}{1 + b_1 k^2}, \quad \xi = \frac{Z_1 Z_2}{hc}\frac{\mu}{137}$$

$$\{a_1 \rightarrow 203.988,\ a_2 \rightarrow 173.887,\ b_1 \rightarrow 4.68665\}$$

Data Analysis (cont'd):



Chi-Squared Test Statistic: 0.0784398
p-value (one-sided): 2.93888 E-35
(Mathematica)



Chi-Squared Test Statistic: 0.943883
p-value (one-sided): 1.09073 E-18
(adjusted)

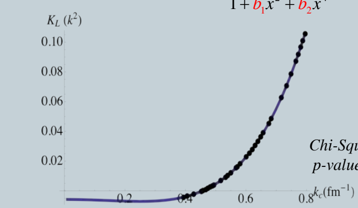
We then changed the values of the fitted parameters to get a function that yielded the highest ANC value:

$a_1 = 242.591$; $a_2 = 264.5$; $b_1 = 8.79$;

Second Interpolation:

Now we find the elastic scattering amplitude $F_l(k)$ using the Padé approximation for the effective range function $K_l(k^2)$:

$$\text{model}(x) = \frac{a_1 + a_2 x^2 + a_3 x^4}{1 + b_1 x^2 + b_2 x^4} = K_l(k^2)$$



Adjusted:
$a_1 = -1/188$;
$a_2 = -.035842$;
$a_3 = 0.3925$;
$b_1 = -.197$;
$b_2 = -0.0958497$;

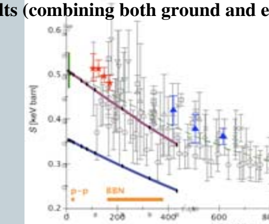
Chi-Squared Test Statistic: 0.4875
p-value (one-sided): 3.0682 E-21
(adjusted)

To get the correct location of the pole we had to take $a_1 = -1/188$ and readjust the fit of $F_l(k)$ to the values listed above. Extrapolating to the pole (E = -1.5866 MeV), we obtain an ANC value of 4.546 $fm^{-1/2}$.

* Repeating the same process with a lower value of $a_1 = -1/152.5$, we get an ANC value of 2.81093.

ANC = 2.81093	Phase Shift (adjusted)	Energy (adjusted)
Chi-Squared Test Statistic	0.0783733	0.000609761
Chi-Squared p-value (one-sided)	2.9006 E-35	3.1255 E-63

Final Results (combining both ground and excited states:



Gyurky, et al. arXiv:nucl-ex/0702003v1 (2007)

Conclusion:

It was demonstrated that, using the analyticity of the elastic scattering S matrix, the experimental elastic scattering amplitude can be interpolated in the physical region and then extrapolated to the pole (in the momentum plane) corresponding to the bound state in order to obtain the ANC. This indirect technique allows one to calculate the astrophysical factor for the peripheral radiative-capture process at astrophysically relevant energies due to the fact that the normalization of the S factor is determined entirely by the ANC.