Midterm EXAM
PHYS 401 (Spring 2012), 03/20/12

Name:

Signature:

*Duration*: 75 minutes
*Show all your work for full/partial credit!*

In taking this exam you confirm to adhere to the Aggie Honor Code:
“An Aggie does not lie, cheat, steal or tolerate those who do.”
1.) **Multiple Choice**

For each statement below, check the correct answer (no reasoning required).

(a) The nuclear decay law, \( \frac{dN}{dt} = -\Gamma N \), is a differential equation of type

- [ ] second-order, partial and linear.
- [ ] first-order, ordinary and linear.
- [ ] first-order, partial and linear.
- [ ] first-order, ordinary and non-linear.

(b) For one-dimensional motion with a non-zero external force and a speed-dependent drag force, the most realistic set-up consists of

- [ ] a constant force at all speeds.
- [ ] a constant power at all speeds.
- [ ] a constant force at small speed and constant power at high speed.
- [ ] a constant power at small speed and constant force at high speed.

(c) The simultaneous Over-Relaxation Method for numerical solution of a finite-boundary problem of the Laplace equation on a \( n \times n \) spatial grid requires a total computing time proportional to

\[ \begin{array}{cccc}
\log(n) & \sqrt{n} & n & n^2 \\
\end{array} \]

(d) The total computing time for a Fast Fourier Transform of a signal in the time domain parametrically depends on the number, \( N \), of time-sampling points as

\[ \begin{array}{cccc}
\log(N) & N & N \log(N) & N^2 \\
\end{array} \]

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2.) Nuclear Decay

Consider the nuclear decay differential equation

\[ \frac{dN}{dt} = -\Gamma N(t) , \]  

where \( \Gamma \) denotes the decay rate.

(a) Derive the finite-difference form of this equation and write it in a form suitable for numerical evaluation. What is the relevant time scale in this problem?

(b) Which mathematical expansion method is equivalent to the finite-difference method of part (a)? Show this by explicitly carrying out this method and comparing it to part (a).

(c) When applying the algorithm of part (a) with a time step of \( \Delta t \), what is the parametric error in each step?

(d) If you carry out the part (a) over \( n \) time steps, what is the total numerical error of your result at the final observation time, \( t_f = n\Delta t \)? Express the total relative error in terms of \( \Delta t \), \( \Gamma \) and \( t_f \).
3.) *Oscillatory Motion* (21 pts.)

Consider a pendulum (length \( l \), mass \( m \)) as a simple harmonic oscillator described by the second-order differential equation

\[
\frac{d^2 \Theta}{dt^2} = -\Omega^2 \Theta(t)
\]

with angular frequency \( \Omega = \sqrt{g/l} \) (period \( T = 2\pi/\Omega \)).

(a) Decompose Eq. (2) into two first order differential equations and write down their discretized form using the Euler Method. Solve for the angular speed, \( \omega_{i+1} \), and angle, \( \Theta_{i+1} \), at time step \( i+1 \) in terms of the values at the preceding step \( i \).

(b) Given the total mechanical energy at time step \( i \),

\[
E_i = \frac{1}{2} ml^2 [\omega_i^2 + \Omega^2 \Theta_i^2]
\]

calculate the energy, \( E_{i+1} \), in the following time step using \( \omega_{i+1} \) and \( \Theta_{i+1} \) from part (a). What problem emerges and how can the Euler Method be amended to avoid this problem?

(c) Suppose now that the description of the pendulum is made more realistic by including damping and nonlinearities. In addition, a variable driving force is switched on. Explain how the concept of the Lyapunov exponent, \( \lambda \), can be used to characterize whether the oscillator’s motion is in a chaotic or regular regime.
4.) *Planetary Motion*  

Use Newton’s 2. law of motion and his general law of gravitation,

$$\vec{F} = -G_N \frac{m_1 m_2}{r^2} \vec{e}_r,$$

(4)

to describe the motion of a planet/asteroid (mass $m_2 = M_P$) around the Sun (mass $m_1 = M_\odot$) in the $x$-$y$-plane. Assume the Sun to be fixed in the origin.

(a) Write down the two second-order differential equations for the planet’s acceleration in $x$ and $y$ coordinates using astronomical units ([length] = average Earth-Sun distance; [time] = 1 year; [mass] = solar mass). Specifically, show that $G_N = 4\pi^2$ in these units (hint: use a circular Earth orbit).

(b) Which algorithm would you use to numerically solve these equations and why?

(c) What is special about the $1/r^2$ dependence of the force law and how can it be tested numerically using planetary motion?
5. Wave Propagation (8+8+5 pts.)

The ideal-wave equation for one-dimensional propagation is given by

\[
\frac{d^2y}{dt^2} = c^2 \frac{d^2y}{dx^2}
\]  

(5)

(a) Derive the symmetric finite-difference form of the second-order time derivative.

(b) Use the finite-difference second-order derivatives to solve the ideal-wave equation as a time evolution at each spatial site. What subtlety (condition) in the choice of \(\Delta t\) and \(\Delta x\) should be taken care of?

(c) Given \(M\) space points, how many initial values of the discretized wave medium (e.g. ideal string) need to be fixed?