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Generalized Weinberg sum rules and the factorization issue of in-medium four-quark condensates

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Electromagnetic Probes of Hot and Dense Matter, ECT*, Trento, Italy, June, 2005

Motivation

- QCD has non-trivial vacuum structure

$\rightsquigarrow$ can be characterized by condensates

- phase transitions (or rapid crossovers) at finite temperatures/densities

$\rightsquigarrow$ changes of condensates $\longrightarrow$ order parameters

- QCD sum rules connect hadronic properties (spectra)
  with quark-gluon properties (perturbation theory + condensates)

$\rightsquigarrow$ interest in four-quark condensates

Contents

- Generalized Weinberg sum rules and order parameters
- Evaluation of four-quark condensates (large N_c)
- Uncertainty estimates
- Low temperatures: pion gas
- Finite baryo-chemical potential (and zero temperature):

  Fermi sphere of nucleons
- Sizable temperatures and chemical potentials: resonance gas, phase transition

Weinberg sum rules

difference of vector ($\mathrm{Im} R^V$) and axial-vector ($\mathrm{Im} R^A$) spectra

(chiral partners, i.e. connected by chiral transformation)

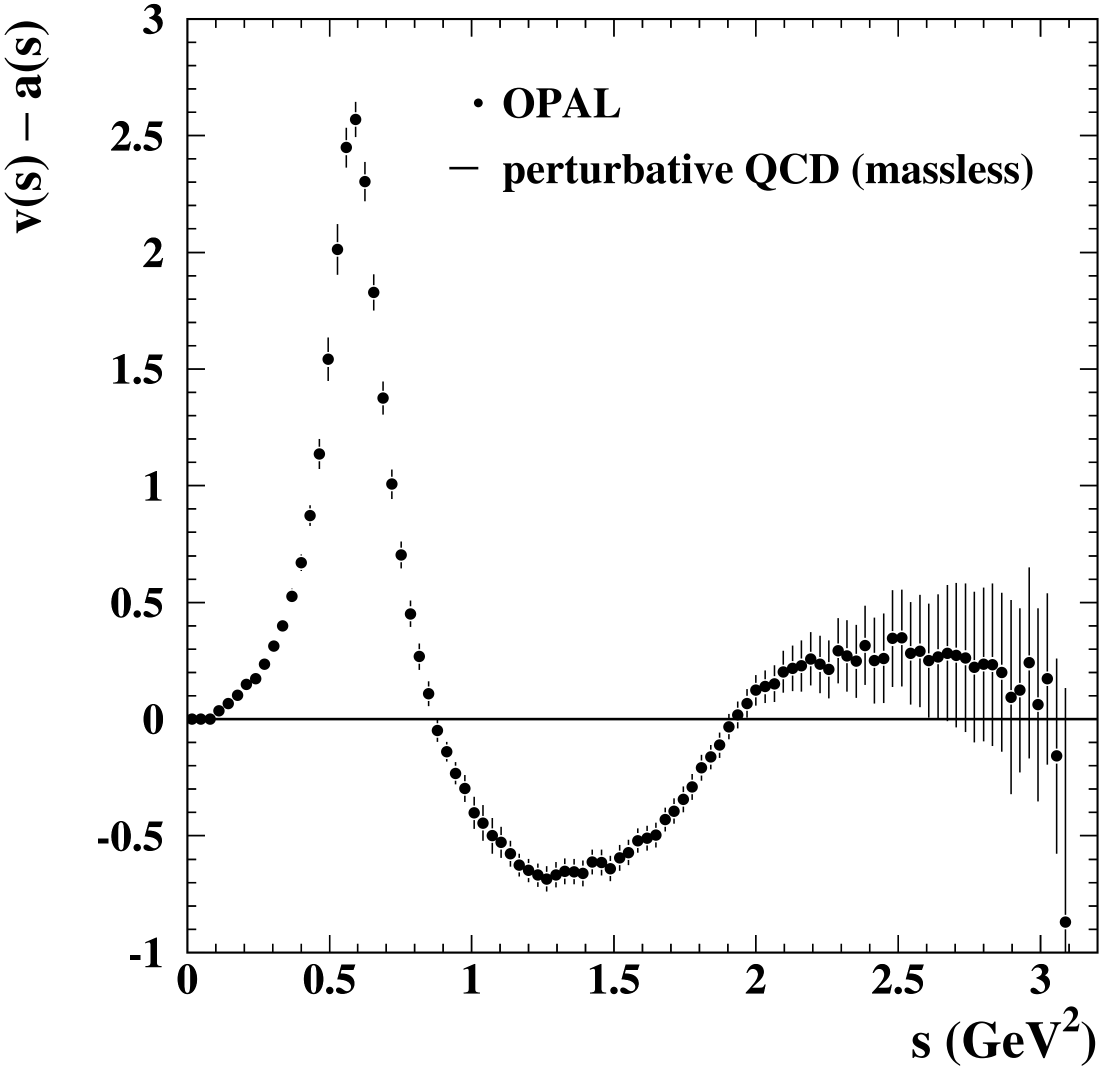
$$\frac{1}{\pi}\int_0^\infty ds\, \left(\mathrm{Im} R^V(s) - \mathrm{Im} R^A(s)\right) = F_\pi^2\,,$$

$$\frac{1}{\pi}\int_0^\infty ds\, s\, \left(\mathrm{Im} R^V(s) - \mathrm{Im} R^A(s)\right) = 0\,,$$

$$\frac{1}{\pi}\int_0^\infty ds\, s^2\, \left(\mathrm{Im} R^V(s) - \mathrm{Im} R^A(s)\right) = -\frac{1}{2}\,\pi\alpha_s \langle \mathcal{O}_{\chi\mathrm{SB}}\rangle\,,$$

with four-quark condensate

$$\langle \mathcal{O}_{\chi\mathrm{SB}}\rangle = \left\langle (\bar{u}\gamma_\mu\gamma_5\lambda^a u - \bar{d}\gamma_\mu\gamma_5\lambda^a d)^2 - (\bar{u}\gamma_\mu\lambda^a u - \bar{d}\gamma_\mu\lambda^a d)^2 \right\rangle$$



Eur. Phys. J.

C7 (1999) 571

- generalization to in-medium situations by Kapusta/Shuryak (1993)

$\hookrightarrow$ $\langle \mathcal{O}_{\chi\mathrm{SB}} \rangle$ is order parameter (somewhat oversimplified ...)

- closer connected to (in principle) measurable quantities (R^V and R^A) than two-quark condensate

$\hookrightarrow$ How to evaluate four-quark condensates in vacuum and medium?

- typical form: $\langle \bar{q}\Gamma\lambda_a q\, \bar{q}\Gamma'\lambda_a q \rangle$
  with spin-flavor matrices Γ and color Gell-Mann matrices λ_a

$\hookrightarrow$ Fierz transformations:
rewrite $\langle \bar{q}\Gamma\lambda_a q\, \bar{q}\Gamma'\lambda_a q \rangle$ as sum of $\langle \bar{q}\Gamma'' q\, \bar{q}\Gamma''' q \rangle$

- $\bar{q}\Gamma q$ connected to hadronic states

Fierz transformation for condensate which appears in generalized Weinberg SR

$$\langle(\bar{u}\gamma_\mu\gamma_5\lambda_a u - \bar{d}\gamma_\mu\gamma_5\lambda_a d)^2 - (\bar{u}\gamma_\mu\lambda_a u - \bar{d}\gamma_\mu\lambda_a d)^2\rangle =$$

$$2\,\langle(\underbrace{\bar{u}u - \bar{d}d}_{\sim\delta^0})^2\rangle + 2\,\langle(\underbrace{\bar{u}u + \bar{d}d}_{\sim f_0\,(\sigma)})^2\rangle - 8\,\langle\underbrace{\bar{u}d}_{\sim\delta^-}\,\bar{d}u\rangle$$

$$2\,\langle(\underbrace{\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d}_{\sim\pi^0})^2\rangle + 2\,\langle(\underbrace{\bar{u}i\gamma_5 u + \bar{d}i\gamma_5 d}_{\sim\eta,\eta'})^2\rangle - 8\,\langle\underbrace{\bar{u}i\gamma_5 d}_{\sim\pi^-}\,\bar{d}i\gamma_5 u\rangle$$

$$-\frac{2}{N_c}\,\langle(\underbrace{\bar{u}\gamma_\mu\gamma_5 u - \bar{d}\gamma_\mu\gamma_5 d}_{\sim\pi^0,a_1^0})^2 - (\underbrace{\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d}_{\sim\rho^0})^2\rangle$$

- vacuum: factorization assumption

$$\langle 0|\bar{q}\Gamma q\,\bar{q}\Gamma' q|0\rangle = \langle 0|\bar{q}\Gamma q|0\rangle\,\langle 0|\bar{q}\Gamma' q|0\rangle \qquad (-\text{ exchange terms})$$

— intuitive (mean-field approximation)

— justified in large-N_c limit (N_c: number of colors) [Novikov et al, 1984]

— in practice: matter of debate

- finite (low) temperature: medium described by pions, use current algebra

$\rightsquigarrow$ finite-temperature four-quark condensates do not factorize [Eletsky 1993, Hatsuda et al 1993]

$\hookrightarrow$ raises doubts whether factorization justified at finite baryon density

strategy: study in-medium four-quark condensates at large N_c

$\longrightarrow$ understand non-factorization at finite temperatures

$\longrightarrow$ figure out what happens at finite densities common practice: evaluate in-medium condensates in linear-density approximation

$$\langle \mathcal{O} \rangle_{\mathrm{med.}} \approx \langle 0|\mathcal{O}|0\rangle + \sum_X \rho_X \langle X|\mathcal{O}|X\rangle$$

— density of medium-constituents ρ_X

— temperature: $X = \pi$, baryon density: $X = N$, more general: resonance gas

physical decomposition (and vacuum factorization):

$$\langle \bar{q}\Gamma q\, \bar{q}\Gamma' q\rangle_{\mathrm{med.}} \approx \langle 0|\bar{q}\Gamma q|0\rangle \, \langle 0|\bar{q}\Gamma' q|0\rangle$$

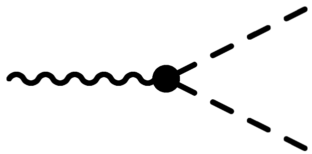
$$+ \rho_X \left(\langle X|\bar{q}\Gamma q|X\rangle \, \langle 0|\bar{q}\Gamma' q|0\rangle + \langle 0|\bar{q}\Gamma q|0\rangle \, \langle X|\bar{q}\Gamma' q|X\rangle \right) \Bigg\{ \text{scatterer + spectator}$$

$$+ \rho_X \left(\langle X|\bar{q}\Gamma q|0\rangle \, \langle 0|\bar{q}\Gamma' q|X\rangle + \langle 0|\bar{q}\Gamma q|X\rangle \, \langle X|\bar{q}\Gamma' q|0\rangle \right) \Bigg\{ \text{annihilation + creation}$$

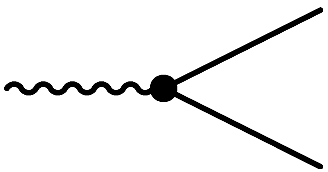
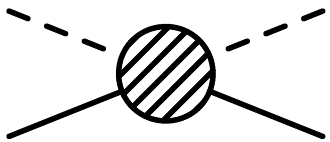
$$+ \rho_X \, \langle X|\bar{q}\Gamma q\, \bar{q}\Gamma' q|X\rangle_{\mathrm{connected}} \qquad \Bigg\{ \text{true scattering}$$

note: last two lines spoil factorization large-N_c scaling ['t Hooft 1974, Witten 1979]

- meson: $q\bar{q}$ state; mass $O(N_c^0)$ 
- baryon: N_c quarks; mass $O(N_c)$
- ratio of quark current $\bar{q}\Gamma q$ and corresponding hadronic field $O(N_c^{1/2})$
- mesonic interactions suppressed:

e.g.  $O(N_c^{-1/2})$

- meson-baryon interactions not suppressed:

 $O(N_c^{1/2})$;  $O(N_c^0)$

$$\langle \bar{q}\Gamma q\, \bar{q}\Gamma' q\rangle_{\rm med.} \approx \langle 0|\bar{q}\Gamma q|0\rangle\, \langle 0|\bar{q}\Gamma' q|0\rangle \qquad O(N_c^2)$$

$$+\rho_X \left(\langle X|\bar{q}\Gamma q|X\rangle\, \langle 0|\bar{q}\Gamma' q|0\rangle + \langle 0|\bar{q}\Gamma q|0\rangle\, \langle X|\bar{q}\Gamma' q|X\rangle\right) \begin{cases} O(N_c) \text{ for } X=\pi \\ O(N_c^2) \text{ for } X=N \end{cases}$$

$$+\rho_X \left(\langle X|\bar{q}\Gamma q|0\rangle\, \langle 0|\bar{q}\Gamma' q|X\rangle + \langle 0|\bar{q}\Gamma q|X\rangle\, \langle X|\bar{q}\Gamma' q|0\rangle\right) \begin{cases} O(N_c) \text{ for } X=\pi \\ \text{vanish for } X=N \end{cases}$$

$$+\rho_X\, \langle X|\bar{q}\Gamma q\, \bar{q}\Gamma' q|X\rangle_{\rm connected} \qquad \begin{cases} O(N_c^0) \text{ for } X=\pi \\ O(N_c) \text{ for } X=N \end{cases}$$

$\hookrightarrow$ no factorization at finite temperature (due to annihilation/creation terms)

$\hookrightarrow$ factorization justified at finite nuclear density (annihilation/creation terms vanish)

Uncertainty estimates

- pion gas: exact calculations possible using current algebra (and vacuum factorization) (Eletsky, Hatsuda/Koike/Lee (1992))

$$\langle \mathcal{O}_{\chi \mathrm{SB}} \rangle_{\text{pionic med.}} = \frac{8(N_c^2 - 1)}{N_c^2} \langle \bar{q} q \rangle_{\text{vac}}^2 \left(1 - \frac{8 \rho_\pi}{3 F_\pi^2} \right)$$

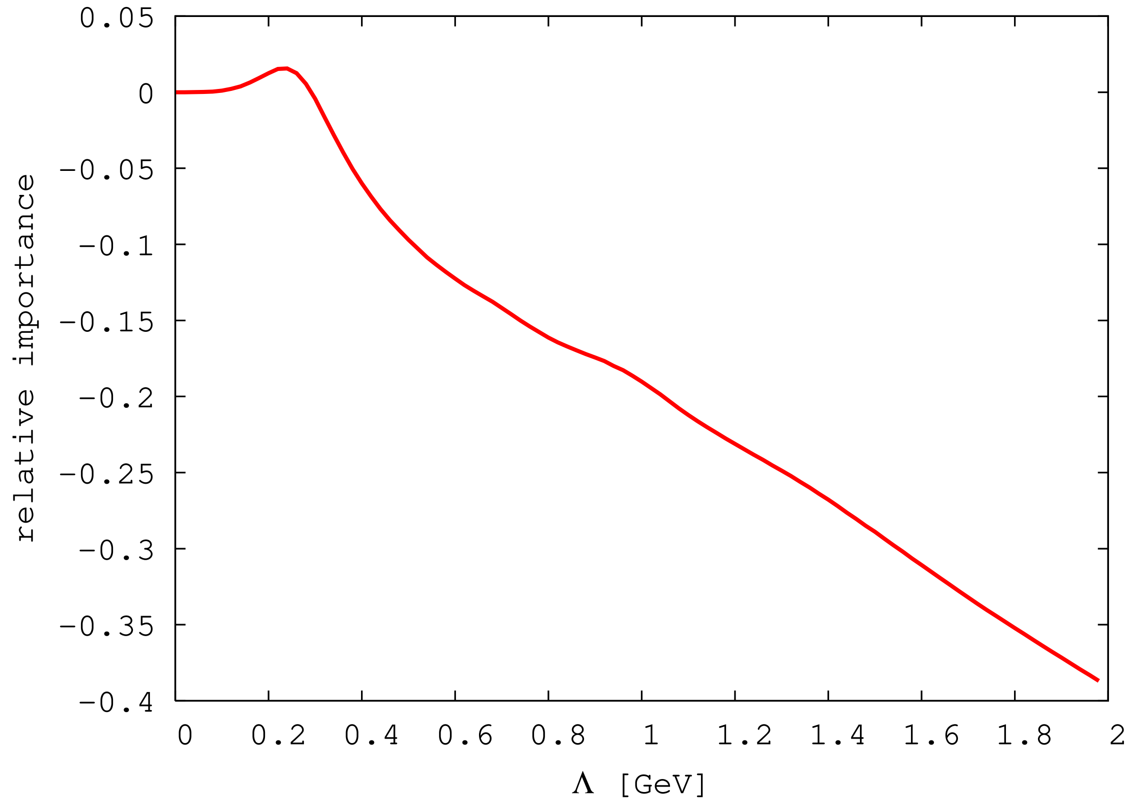
$\hookrightarrow$ $\approx$ 10% correction

- nucleon gas: recall Fierz transformation

$\hookrightarrow$ corrections come from meson-nucleon scattering

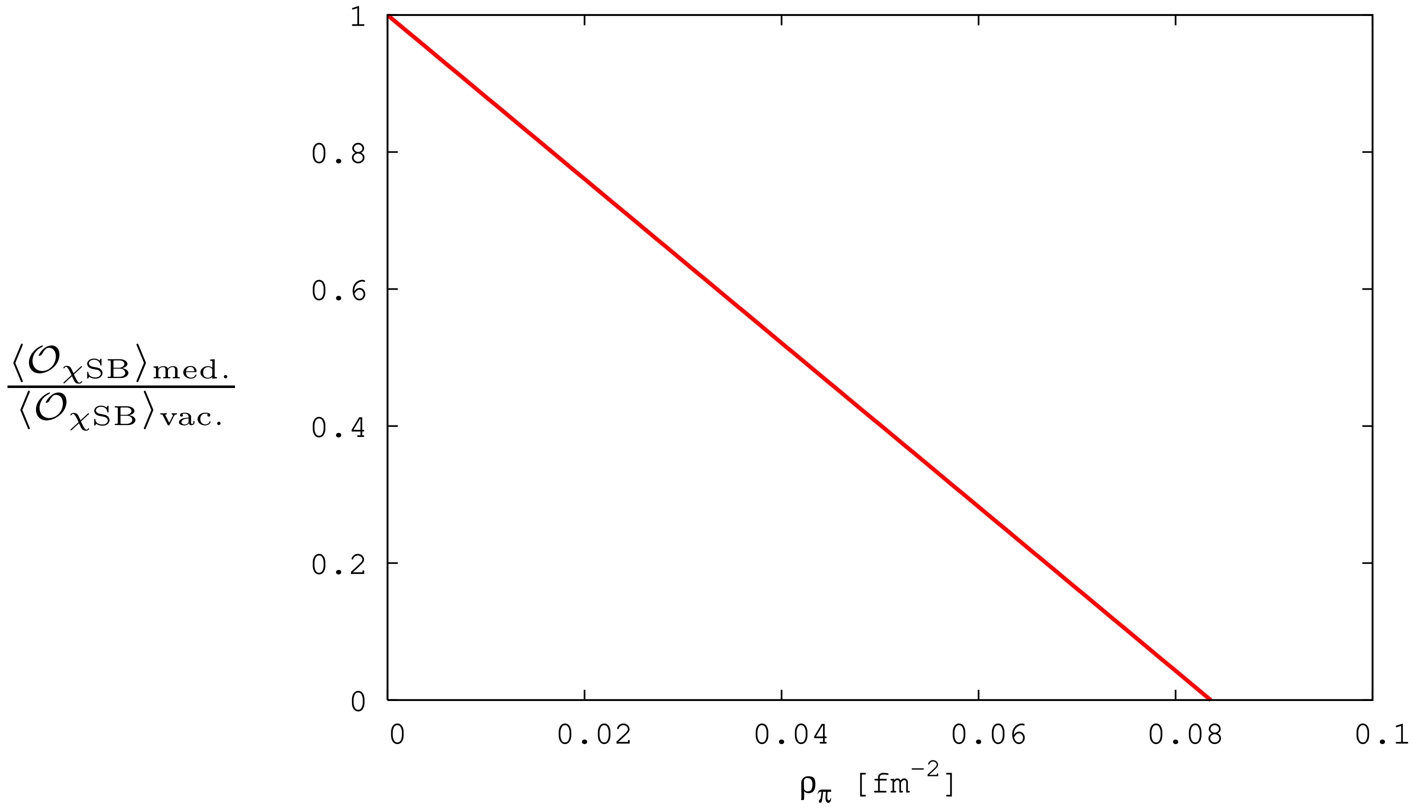
$\hookrightarrow$ can estimate at least pion contribution and compare to factorized part (in-medium generalization of estimate of Shifman et al.)

$\hookrightarrow$ depends on cutoff/separation between non-perturbative and perturbative regime


0.05
0
−0.05
−0.1
−0.15
−0.2
−0.25
−0.3
−0.35
−0.4
relative importance
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Λ [GeV]

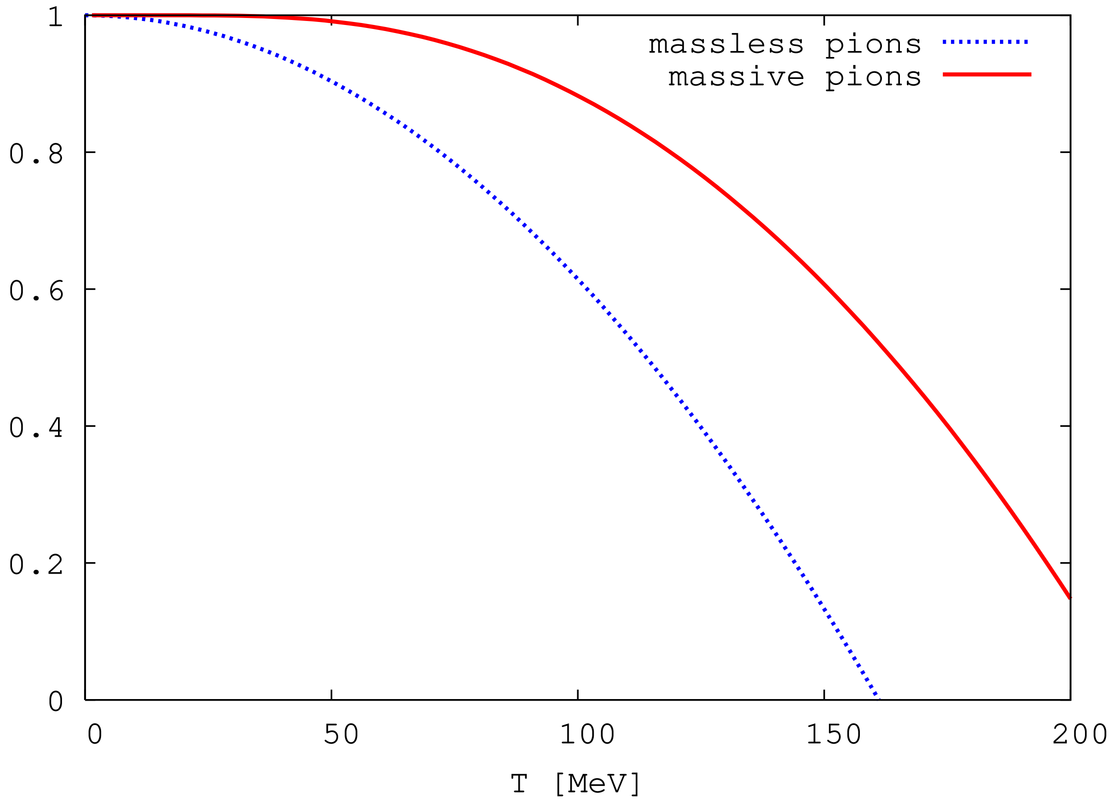
Low temperatures: pion gas

drop of condensate as function of pion density



drop of condensate as function of temperature

$$\frac{\langle \mathcal{O}_{\chi \mathrm{SB}} \rangle_{\mathrm{med.}}}{\langle \mathcal{O}_{\chi \mathrm{SB}} \rangle_{\mathrm{vac.}}}$$



Finite baryo-chemical potential: nucleon gas

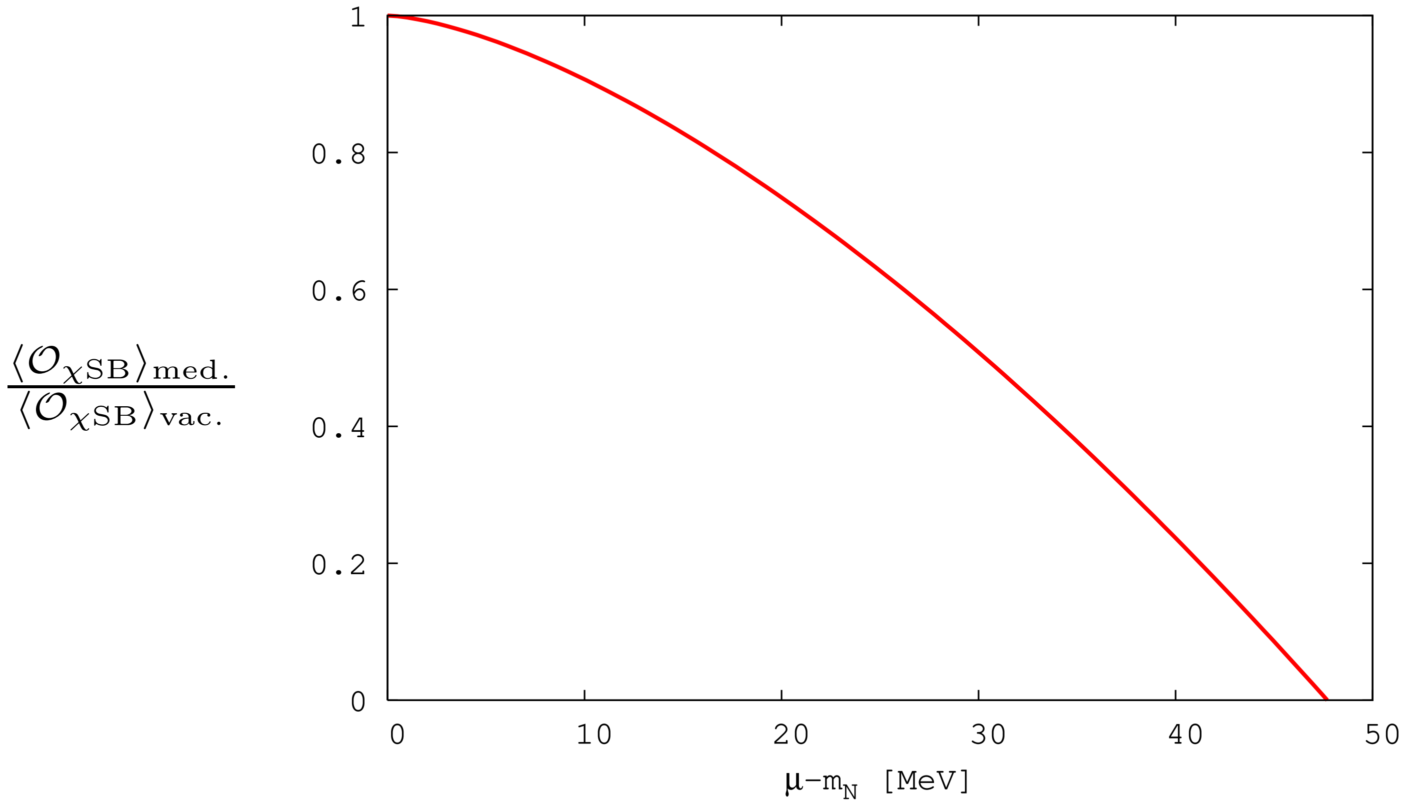
recall: here factorization possible at large N_c

$$\langle \mathcal{O}_{\chi \mathrm{SB}} \rangle_{\text{nuclear med.}} = 8 \langle \bar{q} q \rangle^2_{\text{vac}} \left(1 - \frac{2 \sigma_N \rho_N}{F_\pi^2 M_\pi^2} \right)$$

with nucleon sigma term $\sigma_N \approx$ 45 MeV defined by

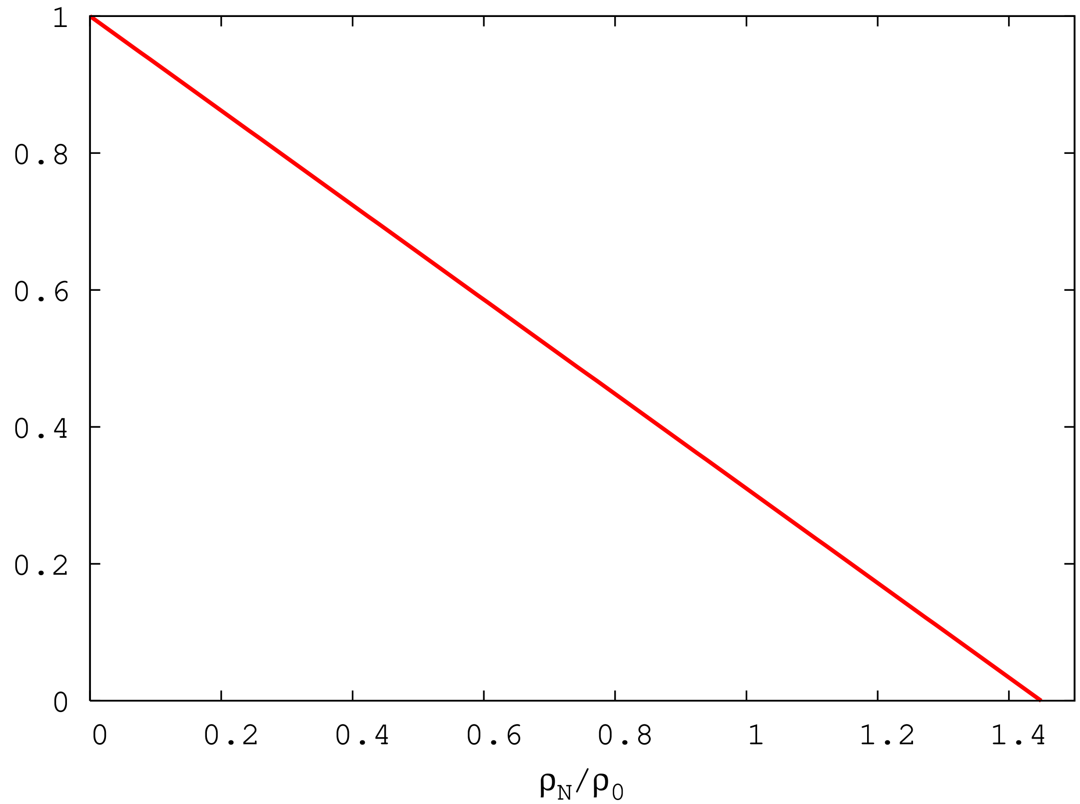
$$\sigma_N = 2 m_q \langle N | \bar{q} q | N \rangle$$

drop of condensate as function of baryo-chemical potential



drop of condensate as function of nucleon density

$$\frac{\langle \mathcal{O}_{\chi \mathrm{SB}} \rangle_{\mathrm{med.}}}{\langle \mathcal{O}_{\chi \mathrm{SB}} \rangle_{\mathrm{vac.}}}$$



presumably effects beyond linear-density approximation become important

at least above ρ_0

Sizable temperatures and chemical potentials: resonance gas

$$\langle \mathcal{O}_{\chi\mathrm{SB}} \rangle_{\mathrm{med.}} \approx \langle 0 | \mathcal{O}_{\chi\mathrm{SB}} | 0 \rangle + \sum_{X = \mathrm{res.}} \rho_X \langle X | \mathcal{O}_{\chi\mathrm{SB}} | X \rangle$$

recall:

- meson contributions subleading in $1/N_c$

$$\langle M | \mathcal{O}_{\chi\mathrm{SB}} | M \rangle = o(N_c)$$

- baryon contributions in leading order, factorization possible

$$\langle B | \mathcal{O}_{\chi\mathrm{SB}} | B \rangle = O(N_c^2)$$

$$\langle \mathcal{O}_{\chi\mathrm{SB}} \rangle_{\mathrm{med.}} \approx 8 \langle \bar{q}q \rangle^2_{\mathrm{vac}} \left(1 - \sum_{X=B} \frac{2\sigma_X \rho_X}{F_\pi^2 M_\pi^2} \right)$$

- with sigma terms defined by

$$\sigma_X = 2m_q \langle X | \bar{q}q | X \rangle$$

- estimate for sigma terms (Gerber/Leutwyler):

  non-relativistic quark model: $\bar{q}q \longrightarrow q^\dagger q$

$$\sigma_X \approx m_q \langle X | u^\dagger u + d^\dagger d | X \rangle = m_q \left(N_c - N_s \right)$$

- underestimates σ_N by about factor of 2

$\rightsquigarrow$ uncertainty

Appropriate for which chemical potentials/temperatures?

$$\rho_B \sim e^{-(M_B-\mu)/T}\,, \qquad \rho_M \sim e^{-M_M/T}$$

- $M_B = O(N_c)$, $M_M = O(1)$

1. $\mu, T = O(1)$

$\hookrightarrow$ $\rho_B \sim e^{-N_c}$ $\quad$ ($\ll 1/N_c$ for large N_c)

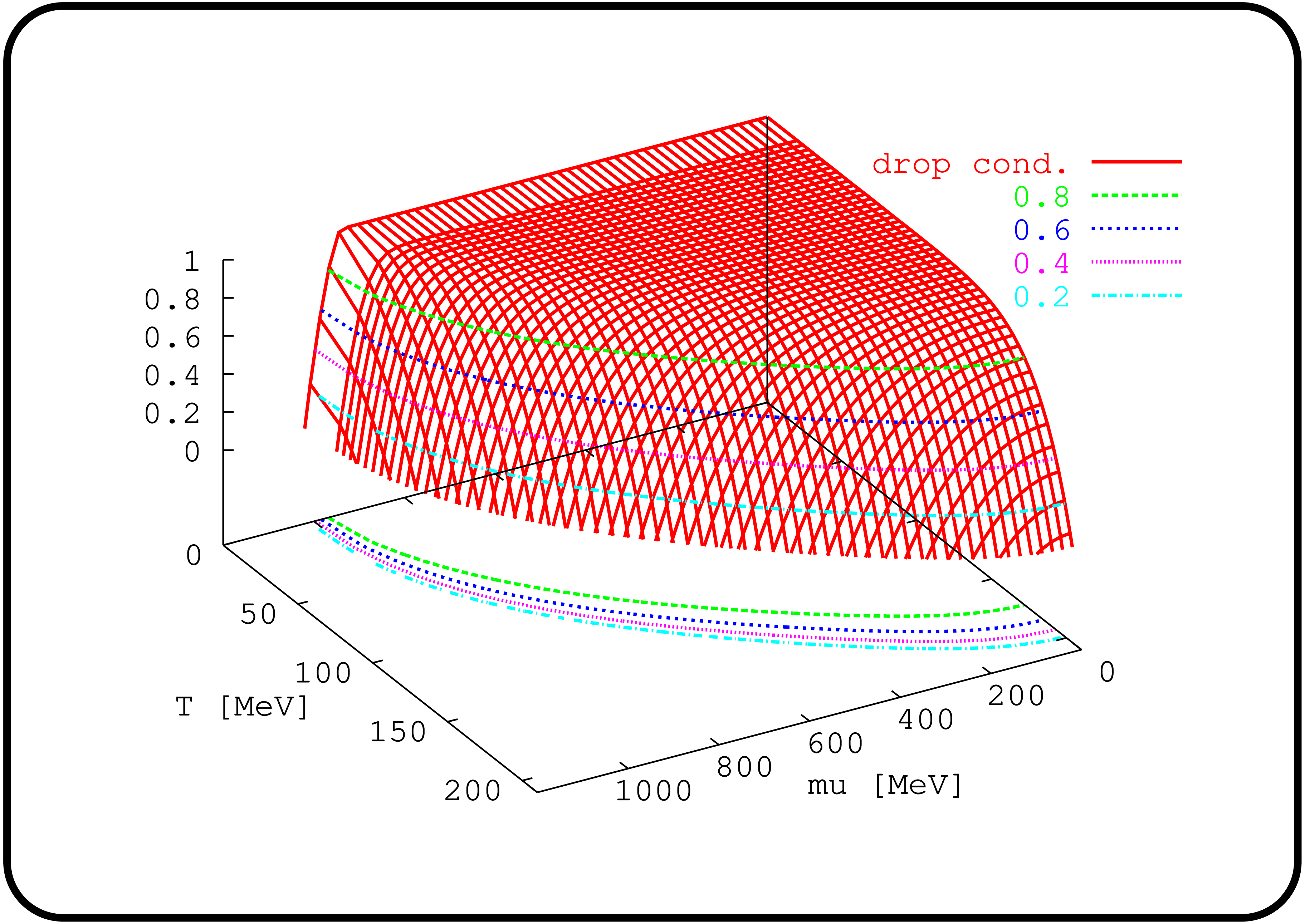
$\hookrightarrow$ baryonic contribution even suppressed!

2. $\mu \sim O(N_c)$, $T = O(1)$

$\hookrightarrow$ $\rho_B = O(1)$, no suppression

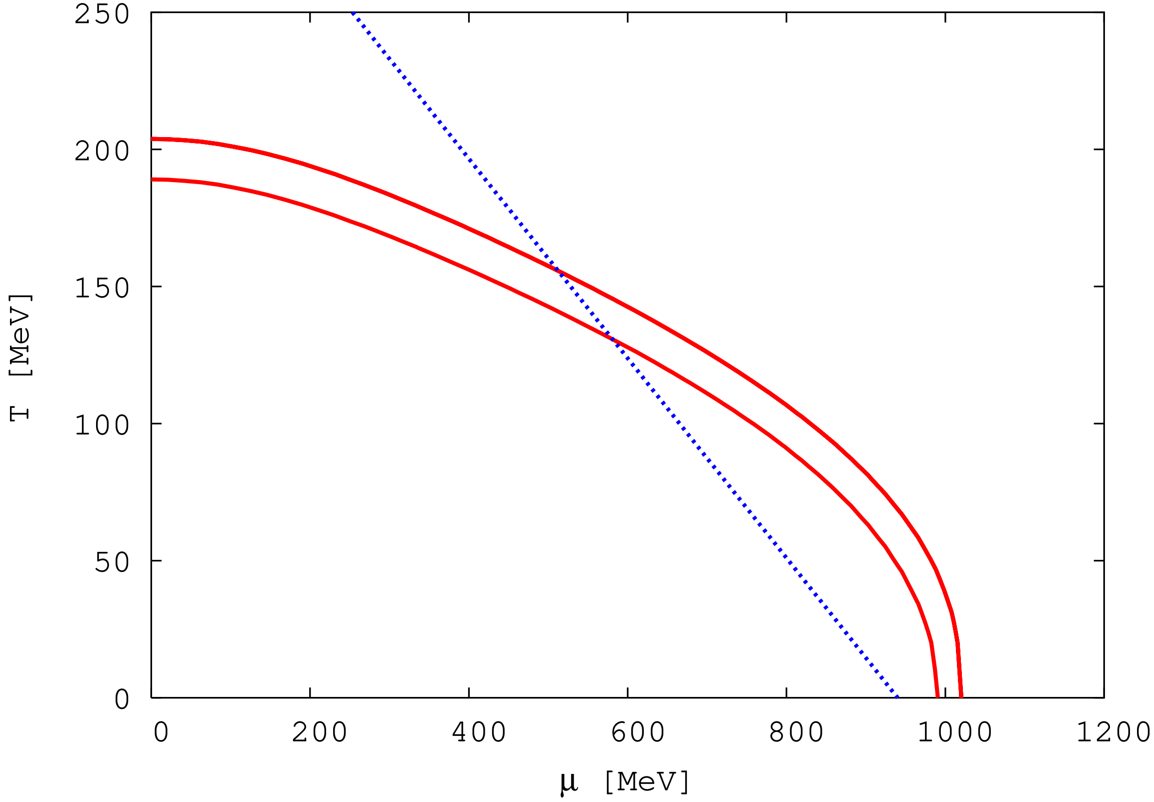
- also without N_c-counting:
  for $\mu > M_N - T$ enhancement of baryons relative to (heavy) mesons

$\hookrightarrow$ appropriate for large chemical potentials and temperatures
(for T too small: no resonances, only nucleons)



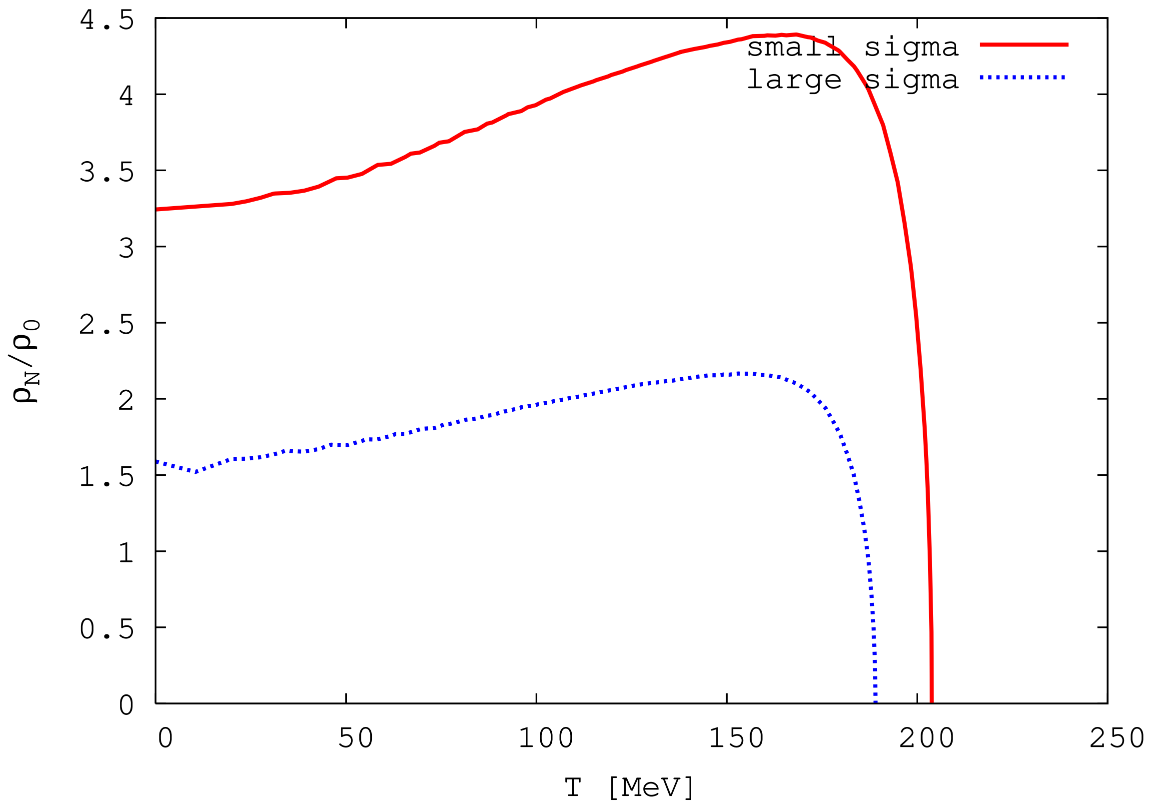
drop cond.
0.8
0.6
0.4
0.2
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
T [MeV]
0
200
400
600
800
1000
mu [MeV]

transition line



reasonable to the right of blue line

net density along transition line



Summary

- consistent picture for large number of colors

  (see also [S.L., hep-ph/0502061, Phys. Lett. B, in press]):

  factorization assumption true for vacuum and finite densities,

  no factorization for finite temperatures

- important ingredient: physical interpretation of different terms

- uncertainty estimates $\rightsquigarrow$ O.K.

- different in-medium situations ...

- resonance gas approximation for sizable temperature and chemical potential

$\hookrightarrow$ phase transition line

exchange terms are always subleading in $1/N_c$:

$$\bar{q}_i \Gamma q_i \, \bar{q}_j \Gamma' q_j \longrightarrow \delta_{ii}\delta_{jj} = N_c^2$$

whereas

$$\bar{q}_i \Gamma q_i \, \bar{q}_j \Gamma' q_j \longrightarrow \delta_{ij}\delta_{ij} = N_c$$

Four-quark condensates in vacuum:

Determination from QCD sum rules possible?

Finite energy sum rules for ρ-meson (somewhat oversimplified ...):

$$\frac{1}{\pi}\int_0^{s_0} ds\,\mathrm{Im}R^{\mathrm{RES}}(s) = \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right)s_0\,,$$

$$\frac{1}{\pi}\int_0^{s_0} ds\,s\,\mathrm{Im}R^{\mathrm{RES}}(s) = \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right)\frac{s_0^2}{2}-\frac{1}{24}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\,,$$

$$\frac{1}{\pi}\int_0^{s_0} ds\,s^2\,\mathrm{Im}R^{\mathrm{RES}}(s) = \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right)\frac{s_0^3}{3}-\frac{112}{81}\pi\bar{\alpha}_s\frac{3}{N_c}\langle\mathcal{O}_{4-\mathrm{q.}}\rangle$$

sum rules have intrinsic uncertainties due to simplifications

$\hookrightarrow$ insert reasonable values and attribute 5% uncertainty (for mass, ...)

$$
\begin{aligned}
\frac{1}{\pi}\int_0^{s_0} ds\,\mathrm{Im}R^{\mathrm{RES}}(s) &= \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right)s_0 \\
17\pm 1.7 &\approx 18\pm 1.8 \\
\frac{1}{\pi}\int_0^{s_0} ds\,s\,\mathrm{Im}R^{\mathrm{RES}}(s) &= \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right)\frac{s_0^2}{2} - \frac{1}{24}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle \\
1.\pm 0.2 &\approx 1.2\pm 0.2 - 0.05 \\
\frac{1}{\pi}\int_0^{s_0} ds\,s^2\,\mathrm{Im}R^{\mathrm{RES}}(s) &= \frac{N_c}{24\pi^2}\left(1+\frac{\bar{\alpha}_s}{\pi}\right)\frac{s_0^3}{3} - \frac{112}{81}\pi\bar{\alpha}_s\frac{3}{N_c}\langle\mathcal{O}_{4-\mathrm{q.}}\rangle \\
6.\pm 2. &\approx 10.\pm 3. - 0.3
\end{aligned}
$$

standard/easy: determination of hadronic parameters from sum rules

complicated: determination of condensates from physical spectrum