Name: PHY625 (Spring 2011), 02/25/11

Homework Assignment #4

(*Due Date*: Mon, Mar. 07, 9:10am, in class)

4.1 *Brueckner-Bethe-Goldstone (BBG) Theory of Nuclear Matter* *(2+2+3+1+2 pts.)*
Starting point is the A-nucleon Schrödinger equation,

$$\hat{H}\ \Psi_{\alpha_1\cdots\alpha_A} = E\ \Psi_{\alpha_1\cdots\alpha_A}\ , \quad \hat{H} = \sum_i^A \hat{T}_i + \sum_{i<j}^A \hat{V}_{ij} \tag{1}$$

with single-nucleon kinetic-energy operators $\hat{T}_i$ and two-nucleon potential operators $\hat{V}_{ij}$.

(a) Show that the plane wave $\phi_\alpha(\vec{r}) = \exp(i\vec{k}_\alpha\cdot\vec{r})/\sqrt{V}$ satisfies the in-medium 1-body Schrödinger equation in coordinate space,

$$T(r)\phi_\alpha(\vec{r}) + \int_{-\infty}^{+\infty} d^3r' U(\vec{r}-\vec{r}\,')\phi_\alpha(\vec{r}\,') = \epsilon_\alpha\phi_\alpha(\vec{r}) \tag{2}$$

with $\epsilon_\alpha = \vec{k}_\alpha^2/2m_N + U_\alpha$, where $U_\alpha = U(k_\alpha)$ denotes the Fourier transform of $U(\vec{r})$. Then use the non-interacting Fermi-gas approximation, $\Psi_{\alpha_1\cdots\alpha_A} = \Phi_{\alpha_1\cdots\alpha_A}$, to show that the nuclear ground-state energy is given by

$$E = \langle\alpha_1\cdots\alpha_A|\hat{H}|\Phi_{\alpha_1\cdots\alpha_A}\rangle = \sum_{k_\alpha<k_F}\left(\frac{k_\alpha^2}{2m_N} + \frac{1}{2}U_\alpha\right)\ , \tag{3}$$

where $|\alpha_1\cdots\alpha_A\rangle \propto \phi_{\alpha_1}\cdots\phi_{\alpha_A}$ with norm $\langle\alpha_2\cdots\alpha_A|\Phi_{\alpha_1\cdots\alpha_A}\rangle = \phi_{\alpha_1}$. Start by multiplying Eq. (1) with $\langle\alpha_2\cdots\alpha_A|$ to identify U_α in terms of V_{ij}. What is the main problem in calculating U_α with the non-interacting Fermi-gas wavefunction?

(b) Defining the 1- and *interacting* 2-nucleon part of the A-body wavefunction as
$\Psi_{\alpha_1} = \langle\alpha_2\cdots\alpha_A|\Psi_{\alpha_1\cdots\alpha_A}\rangle$ and $\Psi_{\alpha_1\alpha_2} = \langle\alpha_3\cdots\alpha_N|\Psi_{\alpha_1\cdots\alpha_A}\rangle$,
with norm $\langle\alpha_1\cdots\alpha_N|\Psi_{\alpha_1\cdots\alpha_A}\rangle = 1$ as before, show that

$$E = \sum_{\alpha_i}\frac{k_{\alpha_i}^2}{2m_N} + \sum_{i<j}\langle\alpha_i\alpha_j|V_{ij}|\Psi_{\alpha_i\alpha_j}\rangle \tag{4}$$

(c) Using a decomposition of the A-body wavefunction as $\Psi_{\alpha_1\cdots\alpha_A} \sim \Psi_{\alpha_1\alpha_2}\Phi_{\alpha_3\cdots\alpha_A}$, schematically derive the 2-body BBG equation,

$$\left[T_1 + T_2 + U_{\alpha_1} + U_{\alpha_2} + \hat{Q}_{\alpha_1\alpha_2}V_{12}\right]\Psi_{\alpha_1\alpha_2} = E_{\alpha_1\alpha_2}\Psi_{\alpha_1\alpha_2} \tag{5}$$

for the 2-body wavefunction and eigenenergies in nuclear matter at fixed density, from the A-body Schrödinger equation (1), with a one-particle "collective potential"

$$U_{\alpha_1}\phi_{\alpha_1} = \langle\alpha_2\cdots\alpha_A|\sum_j G(1j)|\Phi_{\alpha_1\cdots\alpha_A}\rangle\ ,\ \langle\alpha'\beta'|G_{12}|\Phi_{\alpha\beta}\rangle \equiv \langle\alpha'\beta'|\hat{Q}_{\alpha\beta}V_{12}|\Psi_{\alpha\beta}\rangle. \tag{6}$$

(d) Explain why Eqs. (5) and (6) constitute a self-consistency problem.

(e) What is the main effect in the solution of the BBG 2-body wavefunction compared to the noninteracting Fermi gas result? How does this solve the problem (referred to in part (a)) of calculating the interaction energy contribution to the nuclear ground-state energy, E, in the Fermi-gas approximation?