Name: PHY625 (Fall 2014), 10/10/14

Homework Assignment #5

(*Due Date*: Wednesday, Oct. 22, 01:50 pm, in class)

5.1 *Nuclear Shell Model* *(1+1+2 pts.)*
Start from a spherical harmonic oscillator potential for the collective nuclear mean field,

$$U_{HO}(r) = -U_0 + \frac{1}{2} M_N \, \omega^2 \, r^2 \ , \tag{1}$$

inside a nucleus of radius $R_A = 1.2A^{1/3}$, with $U_0 = -50\,\text{MeV}$ and $U(R_A) = 0$.

(a) How large is the oscillator quantum $\hbar\omega$ for a In-115 nucleus?

(b) Given the energy levels of this potential, $E_N = (N+\frac{3}{2})\hbar\omega - U_0$ with $N = 2(n-1)+l$ and $n \geq 1$, $l \geq 0$, determine the proton shell fillings and pertinent totals.

(c) Now include a spin-orbit coupling of type

$$U(r) = U_{HO}(r) - \frac{2}{\hbar^2}\, \alpha \, \vec{L} \cdot \vec{S} \tag{2}$$

where $\vec{S}$ is the nucleon's spin-$\frac{1}{2}$ operator and α carries the dimension of energy. Determine the eigenvalues of $\vec{L} \cdot \vec{S}$ in terms of j, l and s for $j = l \pm \frac{1}{2}$ (and s=$\frac{1}{2}$), and explain how this generates the nuclear magic numbers 2, 8, 20, 28, 50 and 82. What is the approximate magnitude of α for this to work out?

5.2 σ-ω *Model Lagrangian* *(6 pts.)*
The σ-ω (or Walecka) model Lagrangian (using the notation from class) is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m_\sigma^2 \phi^2 - \frac{1}{4}(V_{\mu\nu})^2 + \frac{1}{2} m_\omega^2 V_\mu^2 + \bar{\psi}\, [i\not{\partial} - M_N - g_\omega \not{V} + g_\sigma \phi]\, \psi \ . \tag{3}$$

(a) First consider the free Dirac equation for the nucleon in momentum space,

$$(\not{p} - M_N)\psi = 0 \ . \tag{4}$$

Show that the ansatz $u_s = N[I_2, \vec{\sigma} \cdot \vec{p}/(E_p + M_N)]\chi_s$ for the 4-spinor ψ is a solution, where s=1,2, χ_1=(1,0), χ_2=(0,1), $E_p = (\vec{p}^{\,2} + M_N^2)^{1/2}$, I_2: 2×2 identity matrix, $\vec{\sigma}$: Pauli matrices.

(b) Using the momentum-space expansion of the solutions for free Dirac field operators, ψ and ψ^+ as defined in class, together with the pertinent anti-commutation and orthogonality relations, show that "bare" vacuum energy of Dirac theory is given by

$$H \equiv \int d^3x \mathcal{H} = \sum_{s=1,2} \int d^3p \ E_p \ [b_s^+(p) \ b_s(p) - d_s(p) \ d_s^+(p)] \ . \tag{5}$$

Why is this problematic and how did Dirac solve this problem?

(c) Derive the equations of motion for the σ, ω and nucleon fields in the Walecka model using the Euler-Lagrange equations.