2.1 Liquid Drop Model (LDM) of Nuclei (1+2+1+1+1 pts.)

The empirical Weizsäcker formula for the binding energy of nuclei is given by

\[ E_B = \sum_{i=1}^{5} E_i = -a_1 A + a_2 A^{2/3} + a_3 \frac{Z^2}{A^{1/3}} + a_4 \frac{(A - 2Z)^2}{A} + a_5 \frac{\lambda}{A^{3/4}} \]  

with \( A \): nuclear mass number, \( Z \): nuclear charge (in units of \( e \)), \( a_1 = 15.75 \text{ MeV}, \ a_2 = 17.8 \text{ MeV}, \ a_3 = 0.71 \text{ MeV}, \ a_4 = 23.7 \text{ MeV} \) and \( a_5 = 34 \text{ MeV} \) with \( \lambda = -1,0,1 \) for e-e,e-o,o-o nuclei.

(a) Briefly discuss the physical motivation (\( A \) and \( Z \) dependence) for each term.

(b) Estimate the value of \( a_3 \) theoretically by modeling a nucleus \((A, Z)\) with radius \( R_A = r_0 A^{1/3} \) (\( r_0 = 1.15 \text{ fm} \)) by a uniform spherical charge distribution, \( \rho_c(r) = \rho_{c,0} \Theta(R_A - r) \). First use Poisson’s law, \( \Delta V(r) = -\rho(r) \), to find the static electric potential for \( r < R_A \) (make sure it is continuous with the point-charge potential \( V(r) = Ze/4\pi r \) for \( r > R_A \)). Then calculate the electric potential energy,

\[ U(A, Z) = -\frac{1}{2} \int d^3 r \ V_c(r) \ \rho_c(r) \]  

(c) Derive the value \( Z^* \) for the charge which minimizes the binding energy for fixed \( A \). Plot the resulting valley of stability, \( Z^*(N) \).

(d) Plot \(|E_B(A)|\) using \( Z^*(A) \) from part (c) by subsequently adding the terms of the LDM in numerical order as written above up to (including) \( i = 4 \). In each step determine the \( A \), if any, for which \(|E_B(A)|\) is maximal.

(e) How much energy is released when fissioning \(^{235}U \rightarrow 2^{116}Pd + 3n\)? How many kg of \(^{235}U\) have to be fissioned in 30 days at an output of 1000 MW (neglect any losses)?

2.2 2-Body Density Matrix in Fermi Gas Model (2+1+1 pts.)

The (reduced) 2-body density matrix for a non-interacting Fermi gas of \( A \) nucleons is

\[ \rho^{(2)}(\vec{r}_1, \vec{r}_2) = \frac{1}{A(A - 1)V^2} \sum_{\vec{k}_i, \vec{k}_j} \left[ 1 - \cos(2\vec{k}_i \cdot \vec{r}) \right] \]  

with \( V \): volume, \( \vec{r} \equiv \vec{r}_2 - \vec{r}_1 \) and \( \vec{k}_{ij} = (\vec{k}_i - \vec{k}_j)/2 \).

(a) Use the continuum limit for the momentum sums, \( \sum_{\vec{k}_i} \rightarrow V \int_{0}^{k_F} \frac{d^3 \vec{k}_i}{(2\pi)^3} \) (also implying \( A \rightarrow \infty \)) to show that \( \rho^{(2)}(\vec{r}_1, \vec{r}_2) = g_-(x)/V^2 \) with

\[ g_-(x) = 1 - \left[ \frac{3}{x^2} \left( \frac{\sin x}{x} - \cos x \right) \right]^2 \]  

(b) Using Taylor expansions of the sine and cosine functions, show that \( \lim_{x \rightarrow 0} g_-(x) = 0 \), i.e., \( \rho^{(2)}(\vec{r}_1, \vec{r}_2) \) exhibits “Pauli repulsion”.

(c) Why does the Pauli repulsion in the spatial part of the 2-nucleon density matrix not suffice to stabilize atomic nuclei at a size that increases as \( R_A \propto A^{1/3} \)?