Homework Assignment #3

(3+3+2+1+1 pts.)

3.1 Brueckner-Bethe-Goldstone (BBG) Theory of Nuclear Matter

Starting point is the A-nucleon Schrödinger equation,

\[ \hat{H} \Psi_{\alpha_1 \cdots \alpha_A} = E \Psi_{\alpha_1 \cdots \alpha_A}, \quad \hat{H} = \sum_i \hat{T}_i + \sum_{i<j} \hat{V}_{ij} \]  

(1)

with single-nucleon kinetic-energy operators \( \hat{T}_i \) and two-nucleon potential operators \( \hat{V}_{ij} \) (=\( V(\vec{r}_i, \vec{r}_j) \)) in coordinate space).

(a) Show that the plane wave \( \phi_\alpha(\vec{r}) = \exp(ik_\alpha \cdot \vec{r})/\sqrt{V} \) satisfies the in-medium 1-body Schrödinger equation in coordinate space,

\[ T(r)\phi_\alpha(\vec{r}) + \int_{-\infty}^{+\infty} d^3r'U(\vec{r} - \vec{r}')\phi_\alpha(\vec{r}') = \epsilon_\alpha \phi_\alpha(\vec{r}) \]  

(2)

with \( \epsilon_\alpha = \vec{k}_\alpha^2/2m_N + U_\alpha \), where \( U_\alpha = U(k_\alpha) \) denotes the Fourier transform of \( U(\vec{r}) \). Then use the non-interacting Fermi-gas approximation, \( \Psi_{\alpha_1 \cdots \alpha_A} = \Phi_{\alpha_1 \cdots \alpha_A} \), to show that the nuclear ground-state energy is given by

\[ E = \langle \alpha_1 \cdots \alpha_A | \hat{H} | \Phi_{\alpha_1 \cdots \alpha_A} \rangle = \sum_{\alpha_1 < k_F} \left( \frac{k_\alpha^2}{2m_N} + \frac{1}{2} U_\alpha \right), \]  

(3)

where \( |\alpha_1 \cdots \alpha_A\rangle \propto \phi_{\alpha_1} \cdots \phi_{\alpha_A} \) with norm \( \langle \alpha_2 \cdots \alpha_A| \Phi_{\alpha_1 \cdots \alpha_A} \rangle = \phi_{\alpha_1} \). Start by multiplying Eq. (1) with \( \langle \alpha_2 \cdots \alpha_A| \) to identify \( U_\alpha \) in terms of \( V_{ij} \). What is the main problem in calculating \( U_\alpha \) with the non-interacting Fermi-gas wavefunction?

(b) Defining the 1- and interacting 2-nucleon part of the A-body wavefunction as \( \Psi_{\alpha_1} = \langle \alpha_2 \cdots \alpha_A| \Psi_{\alpha_1 \cdots \alpha_A} \rangle \) and \( \Psi_{\alpha_1 \alpha_2} = \langle \alpha_3 \cdots \alpha_N| \Psi_{\alpha_1 \cdots \alpha_A} \rangle \), with norm \( \langle \alpha_1 \cdots \alpha_N| \Psi_{\alpha_1 \cdots \alpha_A} \rangle = 1 \) as before, show that

\[ E = \sum_{\alpha_i} \frac{k_{\alpha_i}^2}{2m_N} + \sum_{i<j} \langle \alpha_i \alpha_j|V_{ij}|\Psi_{\alpha_i \alpha_j} \rangle. \]  

(4)

(c) Express the potential term in Eq. (4) via the Brueckner G-matrix and relate it to \( U_\alpha \). Write down the Bethe-Goldstone equation for \( \Psi_{\alpha_i \alpha_j} \) and explain the emergence of a self-consistency problem when solving it.

(d) What is the main effect in the solution of the BBG 2-body wavefunction compared to the noninteracting Fermi gas result? How does this solve the problem (referred to in part (a)) of calculating the interaction-energy contribution to the nuclear ground-state energy, \( E \), in the Fermi-gas approximation?

(e) Sketch the results of the BBG theory with realistic NN potentials for the saturation curve of nuclear matter, \( E_N(k_F) \). Does BBG theory fully succeed in describing the empirical saturation regime?