2.1 Liquid Drop Model (LDM) of Nuclei (1+1+2+1+1 pts.)

The empirical Weizsäcker/LDM formula for the binding energy of nuclei is given by

\[ E_B = \sum_{i=1}^{5} E_i = -a_1 A + a_2 A^{2/3} + a_3 \frac{Z^2}{A^{1/3}} + a_4 \frac{(A-2Z)^2}{A} + a_5 \frac{\lambda}{A^{3/4}} \]  

with \( A \): nuclear mass number, \( Z \): nuclear charge (in units of \( e \)) and positive coefficients \( a_i \) (in units of MeV).

(a) Briefly discuss the physical motivation for each term.

(b) Derive the value \( Z^* \) for the charge which minimizes the binding energy for fixed \( A \). Plot the resulting valley of stability, \( Z^*(N) \).

(c) Use plotting software to graph \( E_B(A) \) using \( Z^*(A) \) from part (b) by subsequently adding the terms of the LDM in numerical order as written above up to \( i = 4 \). For each curve read off the value of \( A \), if any, for which \( -E_B(A) \) is maximal. Here and below use \( a_1 = 15.75 \text{ MeV} \), \( a_2 = 17.8 \text{ MeV} \), \( a_3 = 0.71 \text{ MeV} \), \( a_4 = 23.7 \text{ MeV} \), \( a_5 = 34 \text{ MeV} \) and \( \lambda = -1,0,1 \) for e-e, e-o, o-o nuclei, respectively.

(d) Use the LDM to calculate the binding energies of \( ^{235}_{92} \text{U} \), \( ^{144}_{56} \text{Ba} \) and \( ^{89}_{36} \text{Kr} \), and compare to the experimental values that you find on the Internet.

(e) How much energy is set free when fissioning a \( ^{235}_{92} \text{U} \) nucleus into \( ^{144}_{56} \text{Ba} + ^{89}_{36} \text{Kr} + 2n \) (once all are far separated)? Why does \( ^{235}_{92} \text{U} \) not fission spontaneously?

(Hint: compute the Coulomb repulsion between \( ^{144}_{56} \text{Ba} \) and \( ^{89}_{36} \text{Kr} \) when their surfaces just touch, with \( V_C(d) = Z_1 Z_2 \alpha/d \) and nuclear radii \( R_A = r_0 A^{1/3} \) with \( r_0 = 1.1 \text{ fm} \)).

2.2 One- and Two-Body Density Matrix in Fermi Gas Model (FGM) (2+1+1 pts.)

The (reduced) 2-body density matrix in the FGM has been shown to take the form

\[ \rho^{(2)}(\vec{r}_1, \vec{r}_2) = \frac{1}{A(A-1)V^2} \sum_{\vec{k}_1, \vec{k}_2} \left[ 1 - \cos(2\vec{k}_{ij} \cdot \vec{r}) \right] \]  

with \( \vec{r} \equiv \vec{r}_2 - \vec{r}_1 \) and \( \vec{k}_{ij} = (\vec{k}_i - \vec{k}_j)/2 \).

(a) Using the continuum limit for the momentum sums, \( \sum_{\vec{k}} \rightarrow V \int_0^{k_F} \frac{d^3k}{(2\pi)^3} \) (also implying \( A \rightarrow \infty \)), show that \( \rho^{(2)}(\vec{r}_1, \vec{r}_2) = g_-(x)/V^2 \) with

\[ g_-(x) = 1 - \left[ \frac{3}{x^2} \left( \frac{\sin x}{x} - \cos x \right) \right]^2. \]  

(b) Quote the result for \( \rho^{(2)}(\vec{r}_1, \vec{r}_2) \) for a symmetric spatial part of the A-body wave function. Use plotting software to graph both \( g_+(x) \) over a relevant range in \( r \) for \( k_F = 265 \text{ MeV} \). Interpret the behaviors for \( x \rightarrow 0 \).

(c) Discuss the relevance of both \( g_\pm(x) \) in building the total A-nucleon wave functions, i.e., including the spin-isospin parts.