Name: PHY625 (Fall 2008), 11/20/08

Homework Assignment #8

(*Due Date*: Tuesday, December 2, 05:30 pm, in class)

8.1 *Local Gauge Invariance of QCD Lagrangian* *(3 pts.)*

The QCD Lagrangian is defined by

$$\mathcal{L}_{\rm QCD} = \bar{q}\left(i\gamma^\mu D_\mu - \hat{m}_q\right) q - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \tag{1}$$

with the covariant derivative $D_\mu \equiv \partial_\mu + i g_s T_a G^a_\mu$ and the (nonabelian) gluon field strength tensor $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g_s f_{abc} G^b_\mu G^c_\nu$; the generators of $SU(3)$, $T_a = \lambda_a/2$, are defined in terms of the Gellmann matrices $\lambda_{1...8}$, and satisfy the algebra $[T_a, T_b] = i f_{abc} T_c$ (summation over double indices implied). Show that $\mathcal{L}_{\rm QCD}$ is invariant under the following *local* (infinitesimal) phase rotation:

$$\begin{aligned} q(x) &\rightarrow [1 + i\alpha_a(x)\, T_a] q(x) \\ G^a_\mu(x) &\rightarrow G^a_\mu(x) - \frac{1}{g_s}\, \partial_\mu \alpha_a(x) - f_{abc}\, \alpha_b(x)\, G^c_\mu(x) \end{aligned}$$

8.2 *Heavy-Quark Potential and Bottomonium Spectrum* *(4 pts.)*

The static QCD potential between a (heavy) quark and antiquark is well represented by

$$V_{Q\bar{Q}}(r) = -\frac{4}{3}\frac{\alpha_s}{r} + \sigma r \tag{2}$$

with a string tension of $\sigma = 1$ GeV/fm. The ground and first excited state of the Upsilon spectrum (bottom-antibottom bound states) have masses of $M(\Upsilon(1S)) = 9.46$ GeV and $M(\Upsilon(2S)) = 10.02$ GeV, respectively.

(a) Assuming α_s=0.5 and neglecting the linear confining term in the potential, use the hydrogen expression for the bound-state energy to estimate the bottomonium binding energies. Can you find a reasonable value for the bottom-quark mass, m_b, to establish rough agreement with the above masses?

(b) Use the hydrogen-like radii of the Υ states to estimate the correction introduced by the confining term.

(c) Generalize the expression for the electromagnetic hyperfine splitting,

$$\Delta E_{hf} = -\frac{2}{3}\, \vec{\mu}_1 \cdot \vec{\mu}_2\, |\psi(0)|^2 \tag{3}$$

($\vec{\mu}_i = e_i \vec{\sigma}_i / 2m_i$, $|\psi(0)|^2$: onium wave function overlap at the origin), to color charges by replacing $\alpha_{em} \rightarrow \frac{4}{3}\alpha_s$. Prove $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = 2\vec{S}^2 - 3$ and use this relation to estimate the hyperfine splitting between the ground-state bottomonia $\Upsilon(1S)$ (spin S=1) and η_b (spin S=0), as well as between the ground-state charmonia J/ψ (S=1) and η_c (S=0). For the charmonia, use m_c=1.8 GeV and a radius of 0.4 fm, and compare ΔE_{hf} to the measured J/ψ-η_c mass splitting, $M_{J/\psi} - M_{\eta_c}$=(3097−2980) MeV≃120 MeV.

8.3 *Constituent-Quark Mass and Quark Condensate* *(3 pts.)*

In the Nambu Jona-Lasinio model, the gap equation for the constituent light-quark mass is given by

$$m_q^* = m_q + 4N_c N_f G \int \frac{d^3p}{(2\pi)^3} \frac{m_q^*}{(\vec{p}^2 + (m_q^*)^2)^{1/2}} F(p)^2 \tag{4}$$

where $m_q \simeq 5$ MeV is the bare quark mass, G the 4-point q-$\bar{q}$ coupling constant and $F(p) = \Lambda^2/(\Lambda^2 + \vec{p}^2)$ an effective formfactor. Set Λ=0.6 GeV.

(a) Compute numerically the value of the coupling constant, G, for which the constituent quark mass acquires a value of 350 MeV. How large is the quark-antiquark condensate in this case?

(b) For cold quark matter and vanishing bare quark mass, compute and plot $\chi_0(\mu_q) = m_q^*(\mu_q)/2G$, to numerically determine the critical chemical potential and quark density for which the constituent quark mass (and thus the quark condensate) vanishes.